MATH 124

Examination 2h
November 4, 2008

Name ______________________

For a full credit give your answers as exact numbers, not decimal approximations.
Please, absolutely no collaboration or help from others. You can ask me if something is not clear.

1. Use the information given in Problem 1 on Exam 2 to find a rational approximation of the number
$$\cos\left(\frac{11}{7}\right).$$
Your answer should look like $\dfrac{a}{b}$ where a is an integer and b is a four-digit integer.

2. A ball is launched vertically into the air and its distance from the ground (in feet) at any time $t \geq 0$ (t is in seconds) is given by $h(t) = 100(1 - e^{-t})$. The ball is equipped with a remote operated cruise control device (i.e. we can fix the velocity of the ball at any moment).
   (a) Show that $h(t) < 100$ for all $t \geq 0$. That is the ball will never reach the height of 100 feet.

   Now suppose that the ball moves in a vertical pipe which is 100 feet long and which has a gate at the top end.

   (b) The gate will open 8 seconds after the ball has been launched and stay open for a blink of an eye. At which earlier time $t = t_0$ should you fix the velocity of the ball in order for the ball to leave the pipe?
   (c) The gate will open τ (here $\tau > 0$) seconds after the ball has been launched and stay open for a blink of an eye.
      i. Find the formula for the corresponding earlier time $F(\tau)$ such that the ball whose velocity is fixed at the time $F(\tau)$ will leave the pipe at the time τ.
      ii. Taking the setting of this problem into account, answer the following question: Is the function $F(\tau)$ defined for all τ?

   Illustrate with a picture.

3. Let b be a real number, $b > 1$. Consider the exponential function with base b, that is consider the function $f(x) = b^x$. Consider also the inverse function $f^{-1}(x)$. For which base $b > 1$ the graphs of f and its inverse f^{-1} touch? (For example, if $b = e$ than $f(x) = e^x$ and $f^{-1}(x) = \ln x$. Plotting the graphs of these two functions clearly indicates that they do not touch. Hence $b = e$ is not a solution. Try also $b = 2$. Try other values of b. It is essential that you try many examples. Draw some graphs by hand. You must make an important geometric observation about how graphs of f and its inverse relate to each other. Hint: You can find a formula for the inverse function by solving $b^y = x$ for y.)

4. Consider the function $f(x) = x^2\, 2^x$.
   (a) Calculate the first and the second derivative of f.
   (b) Find the value of x for which the first derivative equals 0.
   (c) Find the values of x for which the second derivative equals 0. It is important here to fully simplify the expressions for these points.

   Use the information found in (4a), (4b) and (4c) to identify the maximum intervals where:

   (d) (i) f is increasing; (ii) f is decreasing;
   (e) (i) f is concave up; (ii) f is concave down.

   Use what you found above for the following last item.

   (f) Plot a reasonably detailed and accurate graph of this function. Indicate the information found earlier by using different colors for each of the following four cases: increasing and concave up, increasing and concave down, decreasing and concave up, decreasing and concave down.