MATH 125 Final Examination
March 17, 2009

Name ____________________

Answers which are not clearly supported by the presented work will receive a minimal credit.

1. Find the following integrals:

$$\text{(A)} \quad \int x^2 \sin x \, dx, \qquad \text{(B)} \quad \int e^{\sqrt{x}} \, dx, \qquad \text{(C)} \quad \int \frac{1}{1+\sqrt{x}} \, dx.$$

2. (a) Calculate the exact values of the following improper integrals:

$$\text{(A)} \quad \int_0^{\infty} e^{-x} \, dx, \qquad \text{(B)} \quad \int_0^{1} \bigl(-\ln(x)\bigr) \, dx, \qquad \text{(C)} \quad \int_0^{\infty} x\, e^{-x} \, dx.$$

   Please present your work clearly and in detail.

   (b) Explain with a picture and a calculation a connection between the functions integrated in (A) and in (B). Make sure that this connection supports your calculations in (2a). Also, illustrate with pictures that your answers in (A) and (C) both make sense.

3. Figure 1 below shows a vase obtained by rotating a sine curve with the equation $y = 2 + \sin x$, $0 \le x \le 2\pi$, about the x-axis. Calculate the exact volume of the vase. Hint: The following trigonometric identity might be helpful here: $(\sin x)^2 = (1 - \cos(2x))/2$.

4. Calculate the length of the curve shown in Figure 2. This curve is the graph of the function $f(x) = (2/3)x^{3/2}$ where $0 \le x \le 3$. The figure also includes three light gray line segments. Calculate the lengths of these line segments and compare their lengths to the length of the curve you calculated.

5. Figure 3 shows a curve that I plotted using polar coordinates. The equation that I used is

$$r = f(\theta) = |\theta|, \qquad -\pi \le \theta \le \pi.$$

   Calculate the exact area inside the curve shown in Figure 3. Figure 3 also includes a light gray rectangle. Use this rectangle to verify your answer.

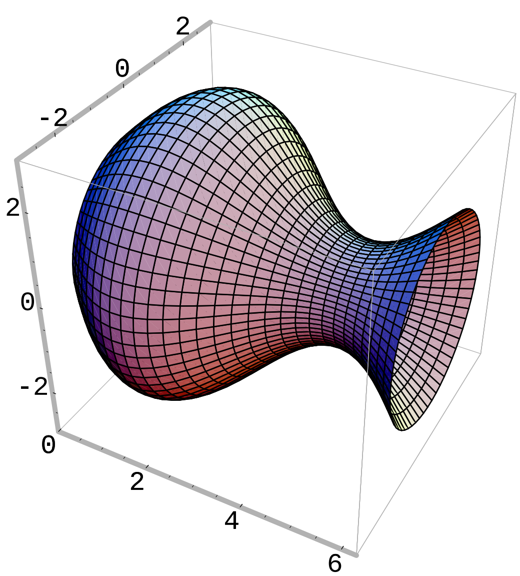


Figure 1: A vase in Pr. 3

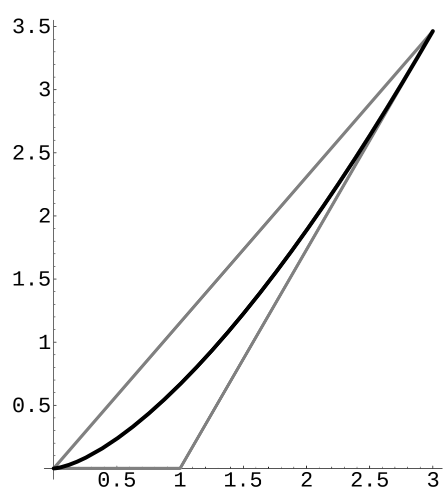


Figure 2: Pr. 4

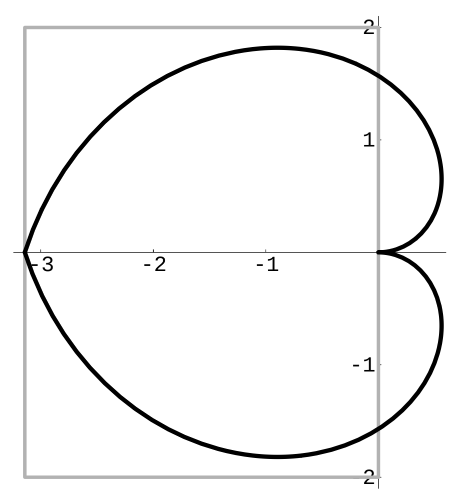


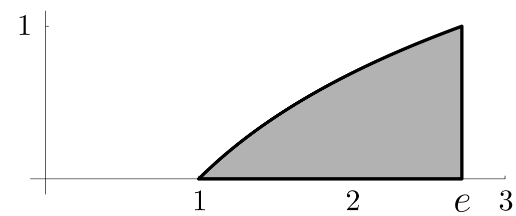
Figure 3: Pr. 5

6. The top of a fence is given by the equation $y = 5 + |\sin(\pi x)|$. What is the average height of this fence between $x = 0$ and $x = 8$? Can you say, without calculating, what is the average height of this fence between $x = 0$ and $x = 20090317$?

7. Figure to the right shows a region bounded by the graph of

$$y = \ln x, \quad 1 \le x \le e,$$

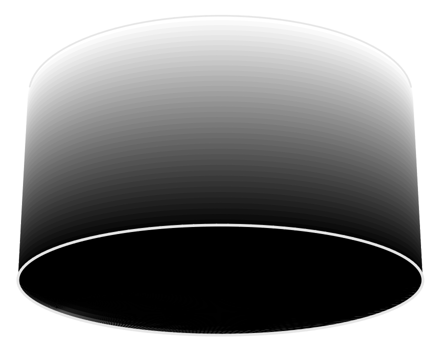
the x-axis and the vertical line $x = e$. This region is made of a material of uniform density 1. Calculate the exact center of mass $(\overline{x}, \overline{y})$ this region.



8. A birthday cake has a shape of a cylinder whose base is a disk with radius 1 and whose height is 1. The cake is made of horizontal layers of different densities. The density at the bottom is 1 and at the top the density is 0. In fact the density of the cake at the height h is given by the formula $\delta(h) = 1 - h$. Here $0 \le h \le 1$.

   (a) Calculate the exact mass of this cake.

   (b) Determine the exact value of the height, call it h_m, which has the property that the mass of the cake below that height and the mass of the cake above that height are the same. That is, if you cut the cake horizontally in two parts at the height h_m then the pieces will have the same mass.



9. The graph of some function f is given in Figure 4 to the right. It is given that the average value of the function f on the interval $0 \le x \le a$ is 1. Consider the following six numbers.

   n1: The value $f(a)$.

   n2: $\int_0^a f(x)\,dx$.

   n3: $\int_0^a f'(x)\,dx$.

   n4: The average value of the rate of change of $f(x)$ on $0 \le x \le a$. (That is the average value of $f'(x)$ on $0 \le x \le a$.)

   n5: The maximum value of $f(x)$ on $0 \le x \le a$.

   n6: The maximum value of $f'(x)$ on $0 \le x \le a$.

   List the given numbers from the smallest to the largest. If some numbers are equal make that clear in your list.

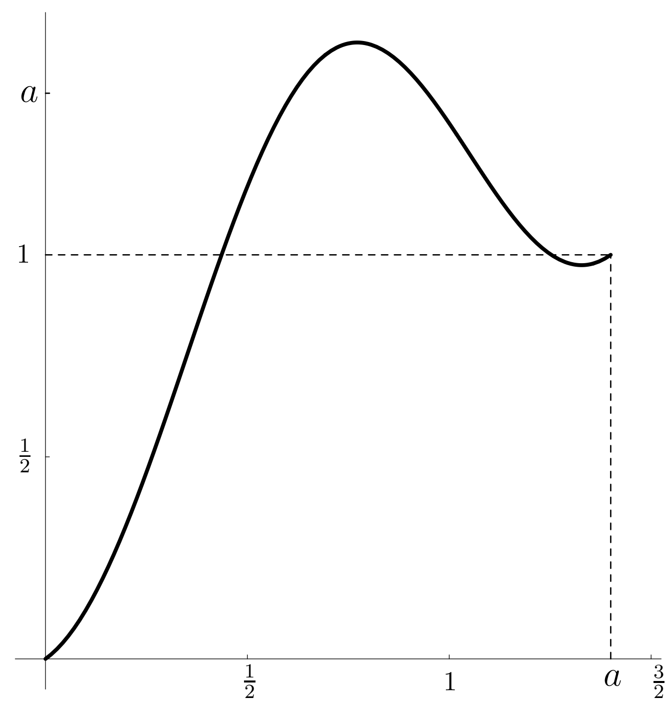


Figure 4: Compare the numbers listed

$\boxed{1}$

① (A) $\int x^2 \sin x\, dx = \left| \begin{array}{l} u = x^2 \\ u' = 2x \\ v' = \sin x \\ v = -\cos x \end{array} \right|$

$= x^2(-\cos x) + \int 2x \cos x\, dx = \left| \begin{array}{l} x = u \\ u' = 1 \\ \cos x = v' \\ v = \sin x \end{array} \right|$

$= \cancel{x2\sin} \; x^2(-\cos x) +$

$+ 2\left(x \sin x - \int 1 \cdot \sin x\, dx\right)$

$= -x^2 \cos x + 2x \sin x + 2\cos x$

verify: $\frac{d}{dx}(-x^2 \cos x + 2x \sin x + 2\cos x)$

$= -\underline{2x\cos x} + x^2 \sin x + \underline{2\sin x} + \underline{2x\cos x} - \underline{2\sin x}$

ok →

(B) $\int e^{\sqrt{x}} dx = \left| \begin{array}{ll} w = \sqrt{x} & \\ x = w^2 & dx = 2w\, dw \\ \frac{dx}{dw} = 2w & \end{array} \right|$

$= \int 2w e^w dw = \left| \begin{array}{l} w = u \\ u' = 1 \\ v' = e^w \\ v = e^w \end{array} \right| =$

$= 2\left(we^w - \int e^w dw\right) = 2we^w - 2e^w$

$= \underbrace{2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}}}$

verify: $\frac{d}{dx}(\quad) = \cancel{\frac{1}{\sqrt{x}} e^{\sqrt{x}}} + \cancel{2\sqrt{x}} e^{\sqrt{x}} \frac{1}{\cancel{2\sqrt{x}}} - \cancel{2e^{\sqrt{x}} \frac{1}{2\sqrt{x}}}$

ok →

① (C)

$$\int \frac{1}{1+\sqrt{x}}\,dx = \left| \begin{array}{l} 1+\sqrt{x} = w \\ \sqrt{x} = w-1 \\ x = (w-1)^2 \\ dx = 2(w-1)\,dw \end{array} \right| \quad \boxed{2}$$

$$= 2\int \frac{1}{w}(w-1)\,dw = 2\int \left(1-\frac{1}{w}\right)dw$$

$$= 2w - 2\ln w$$

$$= \underbrace{2(1+\sqrt{x}) - 2\ln(1+\sqrt{x})}$$

verify $\frac{d}{dx}\left(\quad\right) = \frac{1}{\sqrt{x}} - \cancel{2}\,\frac{1}{1+\sqrt{x}}\cdot\frac{1}{\cancel{2}\sqrt{x}}$

$$= \frac{1}{\sqrt{x}}\left(1 - \frac{1}{1+\sqrt{x}}\right) = \frac{1}{\cancel{\sqrt{x}}}\,\frac{\cancel{x}+\sqrt{x}-\cancel{x}}{1+\sqrt{x}}$$

$$= \frac{1}{1+\sqrt{x}}$$ ok !

(2a)

$\boxed{3}$

$$\int_0^\infty e^{-x}\,dx = \lim_{b\to+\infty}\int_0^b e^{-x}\,dx$$

$$= \lim_{b\to\infty} -e^{-x}\Big|_0^b =$$

$$= \lim_{b\to\infty}\left(-e^{-b}+1\right) = -0+1 = \boxed{1}$$

(b)

$$\int \ln x\,dx = \int 1\cdot \ln x\,dx$$

$$= \left|\begin{array}{l} v' = 1 \\ u = \ln x \\ v = x \\ u' = 1/x \end{array}\right| = x\ln x - \int 1\,dx$$

$$= x\ln x - x$$

$$\int(-\ln x)\,dx = x(1-\ln x).$$

$$\int_0^1(-\ln x)\,dx = \lim_{a\to 0}\int_a^1(-\ln x)\,dx$$

$$= \lim_{a\to 0} x(1-\ln x)\cancel{dx}\Big|_a^1$$

$$= \lim_{a\to 0}\left(0 - a(1-\ln a)\right)$$

$$= \lim_{a\to 0} a(\ln a - 1) = \lim_{a\to 0} a\ln a$$

$$= \cancel{\lim}\ 1$$

② ⓒ $\int_0^\infty x e^{-x}\,dx = \lim_{b\to\infty} \int_0^b x e^{-x}\,dx$ $\boxed{4}$

$$\int x e^{-x}\,dx = \left| \begin{matrix} x = u \\ u' = 1 \\ e^{-x} = v' \\ v = -e^{-x} \end{matrix} \right| = -x e^{-x} + \int e^{-x}\,dx$$

$$= -e^{-x} - x e^{-x}$$

verify: $\cancel{e^{-x}} - \cancel{e^{-x}} + x e^{-x}$ ok

$$\int_0^\infty x e^{-x}\,dx = \lim_{b\to\infty} \left(-e^{-x}(1+x) \Big|_0^b \right) =$$

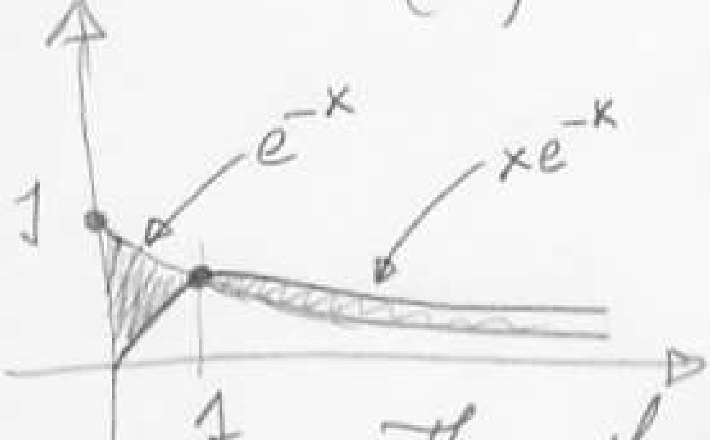
$$= \lim_{b\to\infty} \left(-e^{-b}(1+b) + 1 \right) = 1$$

② ⓑ The function $y = \ln x$ is the inverse of $y = e^{-x}$. To verify this calculate $e^{-(-\ln x)} = e^{\ln x} = x$. Therefore the areas calculated in (A) and (B) are the same since they are symmetric with respect to the line $y = x$.

2(b)

$\boxed{5}$

The integrals in (A) and (B) must be equal.



$(xe^{-x})'$

$e^{-x} - xe^{-x}$

The shaded areas balance out so the integrals (A) and (C) are the same.
This makes sense from the graph.
One should check

$$\int_0^1 (e^{-x} - xe^{-x})\,dx = \int_1^{\infty} (xe^{-x} - e^{-x})\,dx.$$

③ $$\text{Volume} = \int_0^{2\pi} (2+\sin x)^2 \pi \, dx \qquad \boxed{6}$$

$$= \pi \int_0^{2\pi} \left(4 + 4\sin x + (\sin x)^2\right) dx =$$

$$= \pi \left(8\pi + 4(-\cos x)\Big|_0^{2\pi} + \frac{1}{2}\int_0^{2\pi} (1 - \cos 2x)\, dx \right)$$

$$= \pi \left(8\pi + 0 + \frac{1}{2} 2\pi - \left(\frac{1}{4}\sin 2x\right)\Big|_0^{2\pi} \right)$$

$$= \boxed{9\pi^2}$$

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④ $$f(x) = \frac{2}{3} x^{3/2} \qquad f'(x) = \sqrt{x}$$

$$\text{length} = \int_0^3 \sqrt{1 + (f'(x))^2}\, dx$$

$$= \int_0^3 \sqrt{1+x}\, dx = \left| \begin{array}{l} 1+x = w \\ dx = dw \\ \begin{array}{c|c} x & w \\ \hline 0 & 1 \\ 3 & 4 \end{array} \end{array} \right| =$$

$$= \int_1^4 \sqrt{w}\, dw = \frac{2}{3} w^{3/2} \Big|_1^4 =$$

$$= \frac{2}{3}\left(4^{3/2} - 1\right) = \frac{2}{3} \cdot 7 = \boxed{14/3}$$

(4) verify: The shorter line segment is $\sqrt{\underbrace{3^2}_{9}+\underbrace{4\cdot 3}_{12}}=\sqrt{21}$

$\boxed{7}$

$\frac{2}{\cancel{3}}\ \cancel{3}\cdot\sqrt{3}=2\sqrt{3}$

Two line segments are

$$1+\sqrt{4+4\cdot 3}=1+4=5$$

$$\sqrt{21}<\frac{14}{3}<5 \quad \text{TRUE!}$$

true

$$\sqrt{3}\sqrt{7}<\frac{2\cdot 7}{3}\ \rightarrow\ \sqrt{3}<\frac{2}{3}\sqrt{7}\ \rightarrow\ 3\sqrt{3}<2\sqrt{7}$$

$$27<28$$

ok

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(5) The figure is symmetric with respect to the x-axis. So, the area is

$$2\int_0^{\pi}\frac{1}{2}f(\theta)^2\,d\theta=$$

$$=\int_0^{\pi}\theta^2\,d\theta=\frac{1}{3}\theta^3\Big|_0^{\pi}=\frac{\pi^3}{3}$$

rectangle: $4\pi>\pi^3/3$ $\quad 12\pi>\pi^3$

$$\boxed{12>\pi^2}$$

True!

(6) The graph of $y = 5 + |\sin(\pi x)|$ looks like

$\boxed{8}$

denote by A the area $\int_0^1 (5 + |\sin(\pi x)|)\,dx$.

Then $\int_0^8 (5 + |\sin(\pi x)|)\,dx = 8A$

So the average value – height – is A.
The same answer for 20090317.

Calculate A:

$$A = \int_0^1 (5 + |\sin(\pi x)|)\,dx = 5 + \int_0^1 \sin(\pi x)\,dx$$

$$\left|\begin{matrix}\pi x = w \\ dx = \frac{1}{\pi}dw\end{matrix}\right| = 5 + \frac{1}{\pi}\underbrace{\int_0^\pi (\sin w)\,dw}_{= -\cos w\big|_0^\pi = 2} = 5 + \frac{2}{\pi}$$

The average height is $5 + \frac{2}{\pi}$.

(7)

$\boxed{9}$

$$M = \int_1^e \ln x \, dx$$

$$\int \ln x \, dx = x \ln x - x = x(\ln x - 1)$$

$$M = e(\ln e - 1) - 1(\ln 1 - 1) = 1$$

$$\overline{X} = \int_1^e x \ln x \, dx = \begin{vmatrix} v' = x & u = \ln x \\ v = \frac{1}{2}x^2 & u' = 1/x \end{vmatrix} =$$

$$= \frac{1}{2}x^2 \ln x \Big|_1^e - \int_1^e \frac{1}{2}x^2 \frac{1}{x} dx = \frac{1}{2}e^2 - \frac{1}{2}\int_1^e x \, dx$$

$$= \frac{1}{2}e^2 - \frac{1}{4}(e^2 - 1) = \boxed{\frac{1}{4}(e^2 + 1)} \approx 2.097$$

$$\overline{Y} = \int_0^1 y(e - e^y) dy = e\int_0^1 y \, dy - \int_0^1 y e^y dy$$

$$= \frac{1}{2}e - (\underbrace{ye^y - e^y}_{e^y(y-1)}) \Big|_0^1 = \frac{1}{2}e - [0 - 1(0-1)]$$

$$= \frac{1}{2}e - 1 \approx 0.359$$

(8)

$$\text{Mass} = \int_0^1 (1-h)\pi\, dh \qquad \boxed{10}$$

$$= \pi\left(\left(h - \frac{1}{2}h^2\right)\Big|_0^1\right) = \pi/2$$

Mass from $h=0$ to $h=x$

$$\int_0^x (1-h)\pi\, dh = \pi\left(\left(h-\frac{1}{2}h^2\right)\Big|_0^x\right)$$

$$= \pi\left(x - \frac{1}{2}x^2\right)$$

Solve: $\pi\left(1-\frac{1}{2}x^2\right) = \frac{\pi}{4}$

Solve:

$$x - \frac{1}{2}x^2 = \frac{1}{4}$$

or $x^2 - 2x + \frac{1}{2} = 0$

$$x_{1,2} = 1 \pm \sqrt{\frac{1}{2}} = 1 \pm \frac{\sqrt{2}}{2}$$

$$h_m = 1 - \frac{\sqrt{2}}{2} = \frac{2-\sqrt{2}}{2}$$

9

$n1 = 1$ $\boxed{11}$

$n2: \int_0^a f(x) = a$ since

the average value of $f = 1$

$n3 = f(a) - f(0) = 1$

$n4: \frac{1}{a}$ smaler than 1

$n5: > 3/2$

$n6: \approx 2$

$n4 = \frac{1}{a} < 1 = n1 = n3 < n2 = a < n5$
$< n6$

$$\boxed{n4 < n1 = n3 < n2 < n5 < n6}$$