A proof of $\cos\frac{\pi}{5} = \frac{1}{2}\phi = \frac{1+\sqrt{5}}{4}$

The green five-sided polygon is a regular pentagon.

Step 1.

Since the acute angles between the dashed lines are $2\pi/5$, the obtuse angles formed by the green lines are

$$\pi - \frac{2\,\pi}{5} = \frac{3\,\pi}{5}.$$

Consequently, the smaller angles between the blue line and the green lines are

$$\frac{\pi}{2} - \frac{3\,\pi}{10} = \frac{\pi}{5}.$$

Hence, if the green line segments have unit length, then the cosine of $\pi/5$ equals one-half of the blue length.

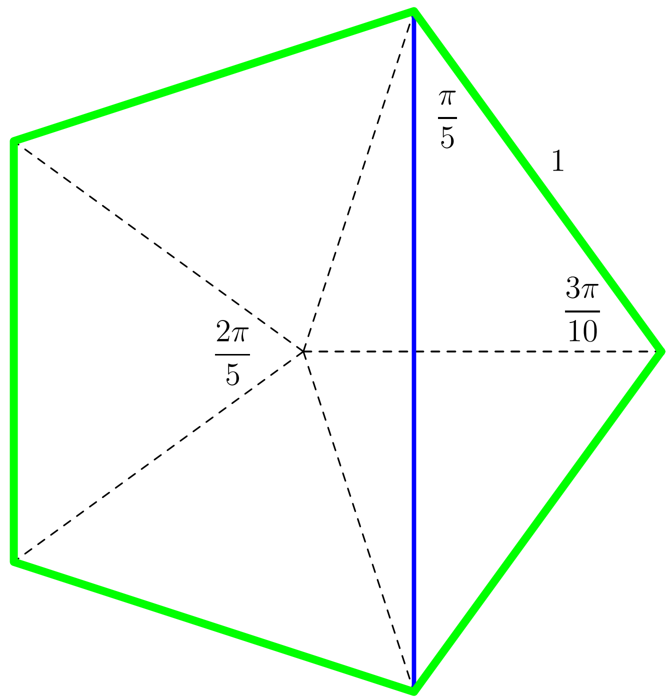

Figure 1: $\cos\frac{\pi}{5}$ equals one-half of the blue length

Step 2.

All the angles in this figure are either $\pi/5, 2\pi/5$, or $3\pi/5$. Therefore the triangles $\triangle ABC$ and $\triangle CAD$ are similar. Let the length of the line segment $\overline{AC}$ be 1 and denote by d the length of the line segment $\overline{AB}$. The similarity of $\triangle ABC$ and $\triangle CAD$ then yields

$$\frac{d}{1} = \frac{1}{d-1}.$$

Thus $d^2 - d - 1 = 0$. The positive solution of this equation is

$$d = \frac{1+\sqrt{5}}{2},$$

the golden ratio ϕ.

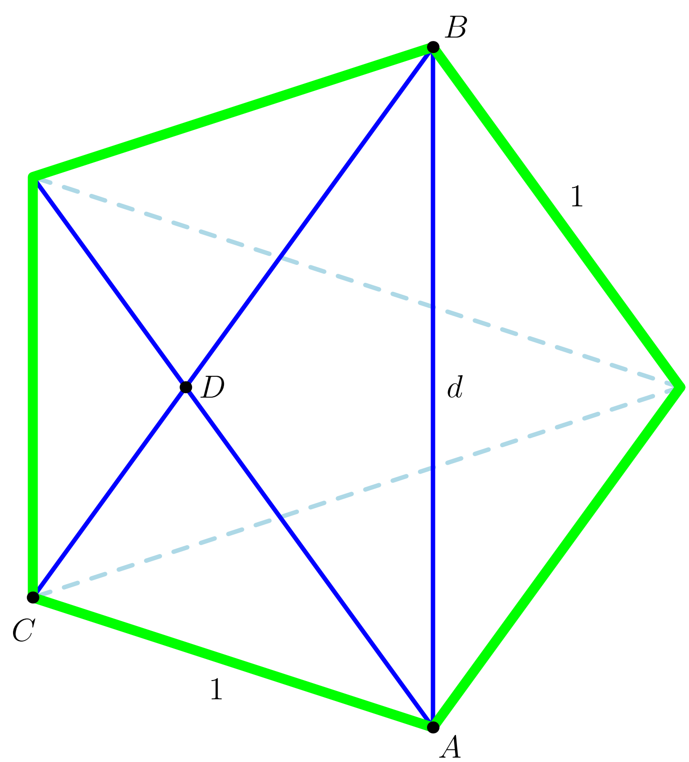

Figure 2: The blue length is the golden ratio