The Chain Rule
and
the Second Fundamental Theorem of Calculus[1]

Problem 1. Find the derivative of the function $G(x) = \int_0^{\sqrt{x}} \sin\big(t^2\big)dt$, $x > 0$.

Solution. Set $F(u) = \int_0^u \sin\big(t^2\big)dt$. Then $F'(u) = \sin(u^2)$. For $x > 0$ we have $F(\sqrt{x}) = G(x)$. Therefore, by the Chain Rule,

$$G'(x) = F'(\sqrt{x})\ \frac{d}{dx}\big(\sqrt{x}\big) = \sin\Big(\big(\sqrt{x}\big)^2\Big)\ \frac{1}{2\sqrt{x}} = \frac{\sin x}{2\sqrt{x}}$$

□

Problem 2. Find the derivative of the function $H(x) = \int_{\sqrt{x}}^{x^2} \sin\big(t^2\big)dt$, $x > 0$.

Solution. Set $F(u) = \int_0^u \sin\big(t^2\big)dt$. Then $F'(u) = \sin(u^2)$. For $x \geq 0$ we have

$$H(x) = F(x^2) - F(\sqrt{x}) = F(x^2) - G(x).$$

Here $G(x)$ was introduced in Problem 1. Now we calculate

$$\frac{d}{dx}\ F(x^2) = F'(x^2)\ 2x = 2x\ \sin(x^4).$$

Hence

$$H'(x) = \frac{d}{dx}\ F(x^2) - G'(x) = 2x\ \sin(x^4) - \frac{\sin x}{2\sqrt{x}}.$$

□

Remark 3. In fact the following "rule" for differentiation holds

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(t)dt = f\big(b(x)\big)\,b'(x) - f\big(a(x)\big)\,a'(x)$$

[1] January 20, 2009 18:59