More integrals for the weekend. This is a good practice for the substitution method of integration. This method has 4 distinct parts:

(1) Define the substitution, express x as a function of the new variable; usually $x =$ (some function of w).

(2) Find the expression for dx as a function of the new variable.

(3) Express the integrand (the function that is being integrated) as a function of the new variable.

(4) Express the new variable as a function of x.

Each of the four parts of the substitution method are clearly distinguished in the examples below. Please follow this style when using substitution.

Integral 1. I first use a magic substitution to find an integral:

$$\int \frac{1}{\sqrt{1+x^2}}\,dx =$$

$$x = \frac{w^2-1}{2\,w} \quad \leftarrow \quad \boxed{\text{Assume that } w>0.}$$

$$\frac{dx}{dw} = \frac{2w\,2w-(w^2-1)\,2}{4\,w^2} = \frac{2\,w^2+2}{4\,w^2} = \frac{w^2+1}{2\,w^2}$$

$$dx = \frac{w^2+1}{2\,w^2}\,dw \quad \leftarrow \quad \boxed{\text{This is the substitution for } dx.}$$

$$\sqrt{1+x^2} = \sqrt{1+\left(\frac{w^2-1}{2\,w}\right)^2} = \sqrt{\frac{4\,w^2+(w^2-1)^2}{4\,w^2}}$$

$$= \sqrt{\frac{4\,w^2+w^4-2\,w^2+1}{4\,w^2}} = \sqrt{\frac{w^4+2\,w^2+1}{4\,w^2}}$$

$$= \sqrt{\left(\frac{w^2+1}{2\,w}\right)^2} = \frac{w^2+1}{2\,w} \quad \leftarrow \quad \boxed{\text{This is a substitution for } \sqrt{1+x^2}.}$$

To find a formula for w in terms of x we solve for w:

$$w^2-2\,x\,w-1=0. \quad \leftarrow \quad \boxed{\text{Here, use the quadratic formula and } w>0}$$

$$w = \frac{2\,x+\sqrt{4\,x^2+4}}{2} = x+\sqrt{x^2+1} \quad \leftarrow \quad \boxed{\text{This is the substitution for } w.}$$

$$= \int \frac{2\,w}{w^2+1}\,\frac{w^2+1}{2\,w^2}\,dw$$

$$= \int \frac{1}{w}\,dw$$

$$= \ln w + C$$

$$= \ln\left(x+\sqrt{1+x^2}\right) + C.$$

Don't forget to celebrate this integral by using the "Onion Rule". But, at this point, be aware of a possible despair!

Now you are asking: Can this integral be solved without this magic substitution? The answer is yes, but one has to use the hyperbolic functions: the hyperbolic sine ("sinh", or briefly "sh"), and the hyperbolic cosine ("cosh", or briefly "ch"). Here are the definitions and some formulas that follow immediately from the definitions:

$$\cosh t = \frac{e^t + e^{-t}}{2}, \tag{1}$$

$$\sinh t = \frac{e^t - e^{-t}}{2} \tag{2}$$

$$(\cosh t)^2 - (\sinh t)^2 = \frac{e^{2t} + 2 + e^{-2t}}{4} - \frac{e^{2t} - 2 + e^{-2t}}{4} = \frac{4}{4} = 1 \tag{3}$$

$$(\cosh t)^2 + (\sinh t)^2 = \frac{e^{2t} + 2 + e^{-2t}}{4} + \frac{e^{2t} - 2 + e^{-2t}}{4} = \frac{e^{2t} + e^{-2t}}{2} = \cosh(2t) \tag{4}$$

$$(\cosh t)(\sinh t) = \frac{e^t + e^{-t}}{2} \, \frac{e^t - e^{-t}}{2} = \frac{1}{2} \, \frac{e^{2t} - e^{-2t}}{2} = \frac{1}{2} \sinh(2t) \tag{5}$$

Adding the formulas (3) and (4) we obtain

$$2\,(\cosh t)^2 = 1 + \cosh(2t),$$

or

$$(\cosh t)^2 = \frac{1}{2}\big(1 + \cosh(2t)\big).$$

The formula (5) can be rewritten as

$$\sinh(2t) = 2\,(\cosh t)(\sinh t).$$

Integral 2. The same integral using a different substitution:

$$\int \frac{1}{\sqrt{1+x^2}}\,dx =$$

$x = \sinh t \quad \leftarrow$ Much simpler substitution inspired by trigonometric substitutions.
$\frac{dx}{dt} = \cosh t$ $dx = (\cosh t)\,dt \quad \leftarrow$ This is the substitution for dx.
$\sqrt{1+x^2} = \sqrt{1+(\sinh t)^2} = \sqrt{(\cosh t)^2}$ $= \cosh t \quad \leftarrow$ This is the substitution for $\sqrt{1+x^2}$.
Next we need a formula for t in terms of x. $t = \sinh^{-1}(x) = \operatorname{asinh}(x) \quad \leftarrow$ This is the the inverse of the hyperbolic sine.

$$= \int \frac{1}{\cosh t}(\cosh t)\,dt$$

$$= \int 1\,dt$$

$$= t + C$$

$$= \sinh^{-1}(x) + C = \operatorname{asinh}(x) + C.$$

Now you are wondering how come that the answers in Integral 1 and Integral 2 are different. It turns out that the answers are same; one can find a formula for $\operatorname{asinh}(x)$.

To get a formula for $\operatorname{asinh}(x)$ we solve for t the equation $x = \sinh t$. By the definition of sinh we have

$$x = \frac{e^t - e^{-t}}{2}.$$

This equation can be solved for t in the following way:

$x = \dfrac{e^t - e^{-t}}{2}$	$\longleftarrow$	multiply by $2\,e^t$.
$2\,x\,e^t = e^{2t} - 1$	$\longleftarrow$	rewrite as a quadratic equation
$\left(e^t\right)^2 - 2\,x\,e^t - 1 = 0$	$\longleftarrow$	solve; taking into account that $e^t > 0$
$e^t = \dfrac{2\,x + \sqrt{4\,x^2 + 4}}{2}$	$\longleftarrow$	simplify
$e^t = x + \sqrt{x^2 + 1}$	$\longleftarrow$	solve for t
$t = \ln\left(x + \sqrt{x^2 + 1}\right)$	$\longleftarrow$	exactly the same expression as in Integral 1

Integral 3. It is surprising that the following integral is harder than Integral 2.

$$\int \sqrt{1+x^2}\,dx = \left|\begin{array}{l} x = \sinh t \quad \longleftarrow \boxed{\text{Much simpler substitution inspired by trigonometric substitutions.}} \\ \hline \dfrac{dx}{dt} = \cosh t \\ dx = (\cosh t)\,dt \quad \longleftarrow \boxed{\text{This is the substitution for } dx.} \\ \hline \sqrt{1+x^2} = \sqrt{1+(\sinh t)^2} = \sqrt{(\cosh t)^2} \\ \qquad\qquad = \cosh t \quad \longleftarrow \boxed{\text{This is the substitution for } \sqrt{1+x^2}.} \\ \hline \text{Next we need a formula for } t \text{ in terms of } x. \\ t = \sinh^{-1}(x) = \operatorname{asinh}(x) \quad \longleftarrow \boxed{\text{This is the the inverse of the hyperbolic sine.}} \end{array}\right|$$

$$\begin{aligned}
&= \int (\cosh t)(\cosh t)\,dt \\
&= \int \left(\frac{e^t+e^{-t}}{2}\right)^2 dt \\
&= \int \frac{e^{2t}+2+e^{-2t}}{4}\,dt \\
&= \frac{1}{4}\left(\int e^{2t}dt + \int 2\,dt + \int e^{-2t}dt\right) \\
&= \frac{1}{4}\left(\frac{1}{2}e^{2t} + 2\,t - \frac{1}{2}e^{-2t}\right) + C \\
&= \frac{1}{2}\,t + \frac{1}{8}\left(e^{2t} - e^{-2t}\right) + C \\
&= \frac{1}{2}\,t + \frac{1}{8}\left(\left(e^{t}\right)^2 - \left(e^{-t}\right)^2\right) + C \quad \longleftarrow \boxed{\text{Here we have a difference of squares.}} \\
&= \frac{1}{2}\,t + \frac{1}{8}\left(\left(e^{t} - e^{-t}\right)\left(e^{t} + e^{-t}\right)\right) + C \quad \longleftarrow \boxed{\text{Remember } a^2-b^2=(a-b)(a+b).} \\
&= \frac{1}{2}\,t + \frac{1}{2}\left(\frac{e^{t} - e^{-t}}{2}\right)\left(\frac{e^{t} + e^{-t}}{2}\right) + C \quad \longleftarrow \boxed{8 = 2\cdot 2\cdot 2.} \\
&= \frac{1}{2}\,t + \frac{1}{2}\,(\sinh t)\,(\cosh t) + C \quad \longleftarrow \boxed{\text{Use } (\cosh t)^2 - (\sinh t)^2 = 1.} \\
&= \frac{1}{2}\,t + \frac{1}{2}\,(\sinh t)\,\sqrt{1+(\sinh t)^2} + C \quad \longleftarrow \boxed{\begin{array}{l}\text{Substitute } t = \operatorname{asinh} x. \\ \text{Remember } \sinh\big(\operatorname{asinh} x\big) = x.\end{array}} \\
&= \frac{1}{2}\left(\operatorname{asinh}(x) + x\,\sqrt{1+x^2}\right) + C.
\end{aligned}$$

Integral 4. The same integral as in Integral 3 calculated using a different method.

$$\int \sqrt{1+x^2}\,dx = \int \sqrt{1+x^2}\,\frac{\sqrt{1+x^2}}{\sqrt{1+x^2}}\,dx$$

$$= \int \frac{1+x^2}{\sqrt{1+x^2}}\,dx$$

$$= \int \frac{1}{\sqrt{1+x^2}}\,dx + \int \frac{x^2}{\sqrt{1+x^2}}\,dx \quad \leftarrow \boxed{\text{The first integral is Integral 1 or 2.}}$$

$$= \ln\left(x+\sqrt{1+x^2}\right) + \int x\,\frac{x}{\sqrt{1+x^2}}\,dx \quad \leftarrow$$

This integral is a good candidate for the integration by parts: $u = x$, $v' = \frac{x}{\sqrt{1+x^2}}$

$$= \left| \begin{array}{l} u(x) = x, \quad v'(x) = \dfrac{x}{\sqrt{1+x^2}} \\ \boxed{\text{Calculating } v(x) \text{ is a good exercise.}} \\ u'(x) = 1, \quad v(x) = \sqrt{1+x^2} \end{array} \right|$$

$$= \ln\left(x+\sqrt{1+x^2}\right) + x\sqrt{1+x^2} - \int \sqrt{1+x^2}\,dx \quad \leftarrow$$

Good news or bad news?
In fact an exelent news!

Now we established

$$\int \sqrt{1+x^2}\,dx = \ln\left(x+\sqrt{1+x^2}\right) + x\sqrt{1+x^2} - \int \sqrt{1+x^2}\,dx.$$

Adding $\int \sqrt{1+x^2}\,dx$ to both sides of this equality we get

$$2\int \sqrt{1+x^2}\,dx = \ln\left(x+\sqrt{1+x^2}\right) + x\sqrt{1+x^2}.$$

Consequently,

$$\int \sqrt{1+x^2}\,dx = \frac{1}{2}\left(\ln\left(x+\sqrt{1+x^2}\right) + x\sqrt{1+x^2}\right) + C.$$