Here are few more integrals. We start with two simple integrals solved by substitution. Then we use algebra, a lots of algebra, to transform complicated looking expressions to sums of these two simple integrals.

Integral 1. In the following integral m and k are real numbers; $m \neq 0$.

$$\int \frac{1}{mx+k}\,dx = \left| \begin{array}{l} w = mx + k \quad \leftarrow \boxed{\text{This is a “natural” substitution.}} \\ \hline \dfrac{dw}{dx} = m \\ dx = \dfrac{1}{m}\,dw \quad \leftarrow \boxed{\text{This is the substitution for } dx.} \end{array} \right|$$

$$\begin{aligned} &= \int \frac{1}{w}\,\frac{1}{m}\,dw \\ &= \frac{1}{m}\int \frac{1}{w}\,dw \\ &= \frac{1}{m}\,\ln|w| + C \\ &= \frac{1}{m}\,\ln(|mx+k|) + C. \end{aligned}$$

Integral 2. In the following integral a is a positive real number. The following integral smells like it is related to $\arctan(x)$. Remember $\frac{d}{dx}\big(\arctan(x)\big) = \frac{1}{1+x^2}$. So my first step is to use algebra get an expression looking like $\frac{1}{1+x^2}$.

$$\int \frac{1}{a^2+x^2}\,dx = \int \frac{1}{a^2\left(1+\frac{x^2}{a^2}\right)}\,dx = \int \frac{1}{a^2}\cdot\frac{1}{1+\frac{x^2}{a^2}}\,dx$$

$$= \frac{1}{a^2}\int \frac{1}{1+\left(\frac{x}{a}\right)^2}\,dx \quad \leftarrow \boxed{\text{This expression suggests a ``natural'' substitution.}}$$

$$= \left| \begin{array}{l} w = \dfrac{x}{a} \quad \leftarrow \boxed{\text{This is a ``natural'' substitution.}} \\ \hline \dfrac{dw}{dx} = \dfrac{1}{a} \\ dx = a\,dw \quad \leftarrow \boxed{\text{This is the substitution for } dx.} \end{array} \right|$$

$$= \frac{1}{a^2}\int \frac{a}{1+w^2}\,dw$$

$$= \frac{1}{a}\int \frac{1}{1+w^2}\,dw$$

$$= \frac{1}{a}\arctan(w) + C$$

$$= \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C.$$

Integral 3. Here is another type of integral that leads to arctan after a substitution.

$$
\begin{aligned}
\int \frac{1}{5-4x+x^2}\,dx &= \int \frac{1}{1+4-4x+x^2}\,dx \\
&= \int \frac{1}{1+(x-2)^2}\,dx \quad \leftarrow \boxed{\text{This expression suggests a ``natural'' substitution.}} \\
&= \left|\begin{array}{l} w = x-2 \quad \leftarrow \boxed{\text{This is a ``natural'' substitution.}} \\ \hline \dfrac{dw}{dx} = 1 \\ dx = dw \quad \leftarrow \boxed{\text{This is the substitution for } dx.} \end{array}\right| \\
&= \int \frac{1}{1+w^2}\,dw \\
&= \arctan(w) + C \\
&= \arctan(x-2) + C.
\end{aligned}
$$

Integral 4. The following integral is very similar to what we did in class today.

$$\int \frac{x^2+2x-3}{x^3-7x-6}\,dx.$$

But, the numerator is definitely not the derivative of the denominator, so the cheap trick of substitution $w = x^3 - 7x - 6$ does not work here.

To use partial fractions we need the roots of $x^3 - 7x - 6 = 0$. As in the textbook, I will give the roots:

$$x^3 - 7x - 6 = (x+1)(x+2)(x-3).$$

Now we can look for A_1, A_2, A_3 such that

$$\frac{x^2+2x-3}{x^3-7x-6} = \frac{A_1}{x+1} + \frac{A_2}{x+2} + \frac{A_3}{x-3}.$$

To determine A_1, A_2, A_3 we write three fractions with a common denominator

$$\frac{x^2+2x-3}{x^3-7x-6} = \frac{A_1(x+2)(x-3) + A_2(x+1)(x-3) + A_3(x+1)(x+2)}{(x+1)(x+2)(x-3)}.$$

Since the denominators of the last to fractions are identical, for the equality to hold, the numerators must be identical as well:

$$x^2+2x-3 = A_1(x+2)(x-3) + A_2(x+1)(x-3) + A_3(x+1)(x+2).$$

Here we have two quadratic expressions. For these two quadratic expressions to be identical they must have identical coefficients with x^2, x and the constant coefficient. The coefficients of $x^2 + 2x - 3$ are easy to see, but the coefficients of

$$A_1(x+2)(x-3) + A_2(x+1)(x-3) + A_3(x+1)(x+2)$$

are somewhat disguised. For example the coefficient with x^2 is $A_1 + A_2 + A_3$.

Rather than determining the coefficient with x and the constant coefficient, I will use Bryce's stated in class today. His idea was to look at roots of the quadratics. Since the roots of $x^2 + 2x - 3$ are -3 and 1, we must have

$$A_1((-3)+2)((-3)-3) + A_2((-3)+1)((-3)-3) + A_3((-3)+1)((-3)+2) = 0,$$
$$A_1(1+2)(1-3) + A_2(1+1)(1-3) + A_3(1+1)(1+2) = 0.$$

That is

$$6A_1 + 12A_2 + 2A_3 = 0,$$
$$-6A_1 - 4A_2 + 6A_3 = 0.$$

In addition to these two equations the coefficients A_1, A_2, A_3 must satisfy

$$A_1 + A_2 + A_3 = 1$$

We rewrite these three equations as

$$\begin{aligned} A_1 + A_2 + A_3 &= 1, \\ 3A_1 + 6A_2 + A_3 &= 0, \\ -3A_1 - 2A_2 + 3A_3 &= 0. \end{aligned}$$

Next we do two operations. First, multiply the first equation by 3 and add it to the third equation, second add the second and the third equation, to obtain:

$$\begin{aligned} A_2 + 6A_3 &= 3, \\ 4A_2 + 4A_3 &= 0. \end{aligned}$$

From the last equation $A_2 = -A_3$, and substituting into the preceding equation we get $5A_3 = 3$, so $A_3 = 3/5, A_2 = -3/5$. Now A_1 is easily calculated, $A_1 = 1$.

Therefore

$$\frac{x^2 + 2x - 3}{x^3 - 7x - 6} = \frac{1}{x+1} - \frac{3}{5}\frac{1}{x+2} + \frac{3}{5}\frac{1}{x-3}.$$

Consequently

$$\int \frac{x^2 + 2x - 3}{x^3 - 7x - 6}\,dx = \int \frac{1}{x+1}\,dx - \frac{3}{5}\int \frac{1}{x+2}\,dx + \frac{3}{5}\int \frac{1}{x-3}\,dx.$$

Now we use Integral 1 with $m = 1$ and an appropriate k. We get

$$\begin{aligned} \int \frac{1}{x+1}\,dx &= \ln|x+1| + C, \\ \int \frac{1}{x+2}\,dx &= \ln|x+2| + C, \\ \int \frac{1}{x-3}\,dx &= \ln|x-3| + C. \end{aligned}$$

Finally, we have the integral, (I simplify the expression involving three logarithms to just one logarithm as an exercise in logarithm identities)

$$\begin{aligned} \int \frac{x^2 + 2x - 3}{x^3 - 7x - 6}\,dx &= \ln|x+1| - \frac{3}{5}\ln|x+2| + \frac{3}{5}\ln|x-3| + C \\ &= \ln|x+1| + \ln\big(|x+2|\big)^{-3/5} + \ln\big(|x-3|\big)^{3/5} + C \\ &= \ln|x+1| + \ln\left(\frac{1}{|x+2|^{3/5}}\right) + \ln\big(|x-3|^{3/5}\big) + C \end{aligned}$$

$$= \ln\left(|x+1|\frac{1}{|x+2|^{3/5}}|x-3|^{3/5}\right) + C$$

$$= \ln\left(|x+1|\left(\frac{|x-3|}{|x+2|}\right)^{3/5}\right) + C$$