

GIVE DETAILED EXPLANATIONS FOR YOUR ANSWERS.

On in class exams I assign four problems. Each is worth 25 points.

I try to assign problems from different topics that we covered.

Below are several problems to help you get used to my style of exam questions.

1. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{v}_4 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_5 = \begin{bmatrix} -1 \\ 1 \\ 4 \\ 4 \end{bmatrix}$ be given vectors in \mathbb{R}^4 .

- Row reduce the 4×5 matrix whose columns are the given vectors. Use the reduced row echelon form to answer the questions below.
- Are vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5$ linearly independent? Give justification for your answer based on the definition. Be specific!
- Find a basis for $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$. What is the dimension of this space? Do vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5$ span \mathbb{R}^4 . Again, be specific!
- What is the dimension of $V = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$? What is the dimension of $W = \text{span}\{\mathbf{v}_4, \mathbf{v}_5\}$? Find a nonzero vector \mathbf{u} which belongs to both subspace V and W . Be specific! Give \mathbf{u} as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and of $\mathbf{v}_4, \mathbf{v}_5$. A complete answer to (1b) can help here.

2. Let $A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -3 & 5 \\ 1 & 0 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 & -1 \\ -2 & -1 & 0 \\ -7 & 4 & 3 \end{bmatrix}$. Determine whether the matrices A and

B have the same column space. (*Important Note:* If you claim that A and B have the same column space justify your claim by calculations. If you claim that the matrices do not have the same column space give a specific vector which is in one column space but not in the other.)

3. (a) Let $\mathbf{v}_1, \dots, \mathbf{v}_m$ be vectors in a vector space \mathcal{V} . State the definition of linear independence for the vectors $\mathbf{v}_1, \dots, \mathbf{v}_m$

The rest of this problem is about the space \mathbb{P}_2 of all polynomials of degree at most 2.

- Consider the polynomials $\mathbf{q}_1(t) = t$, $\mathbf{q}_2(t) = 1 - t^2$, $\mathbf{q}_3(t) = 1 + t^2$ in \mathbb{P}_2 . Are these polynomials linearly independent in \mathbb{P}_2 ? Why or why not?
- Let \mathcal{H} be the set of all polynomials \mathbf{p} in \mathbb{P}_2 such that $\mathbf{p}(-1) = \mathbf{p}(1)$. Show that \mathcal{H} is a subspace of \mathbb{P}_2 .
- Find a basis for \mathcal{H} .

4. Suppose that A is 12×17 matrix and let $S : \mathbb{R}^{17} \rightarrow \mathbb{R}^{12}$ be a linear transformation given by $S(\mathbf{x}) = A\mathbf{x}$.

Suppose that B is 17×12 matrix and let $T : \mathbb{R}^{12} \rightarrow \mathbb{R}^{17}$ be a linear transformation given by $T(\mathbf{x}) = B\mathbf{x}$.

- Can S be one-to-one? Why or why not? Can S be onto? Why or why not? Is there a connection between S being one-to-one or onto and $\text{Nul } A$?

- (b) Can T be one-to-one? Why or why not? Can T be onto? Why or why not? Is there a connection between T being one-to-one or onto and $\text{Nul } B$?
- (c) If $\mathbf{b} \in \mathbb{R}^{12}$ is such that $A\mathbf{x} = \mathbf{b}$ has no solution, what can you conclude about the dimension of $\text{Nul } A$? (Give your answer in the form: $?? \leq \dim \text{Nul } A \leq ??$, where $??$ stand for a specific integer.)
- (d) If $\mathbf{b} \in \mathbb{R}^{17}$ is such that $B\mathbf{x} = \mathbf{b}$ has no solution, what can you conclude about the dimension of $\text{Nul } B$? (Give your answer in the form: $?? \leq \dim \text{Nul } B \leq ??$, where $??$ stand for a specific integer.)
- (e) Which of the matrices AB , BA is defined? Which of these matrices could be invertible? Which of these matrices is never invertible? Explain your answers.
5. In this problem we consider the vector space \mathbb{P}_2 of all polynomials of degree at most 2. Recall that the standard basis for \mathbb{P}_2 is $\mathcal{S} = \{1, t, t^2\}$.
- (a) Consider the polynomials $\mathbf{q}_1(t) = 1 - t^2$, $\mathbf{q}_2(t) = t$, $\mathbf{q}_3(t) = 1 + t^2$ in \mathbb{P}_2 . Prove that $\mathcal{B} = \{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$ is a basis for \mathbb{P}_2 .
- (b) Find $P_{\mathcal{B} \leftarrow \mathcal{S}}$.
- (c) Find the basis \mathcal{C} for \mathbb{P}_2 if $P_{\mathcal{C} \leftarrow \mathcal{B}} = A$, where $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix}$.

6. Consider the matrix $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & 2 & 5 \end{bmatrix}$.

- (a) Find all eigenvalues of A .
- (b) For each eigenvalue find a corresponding eigenvector.

7. Consider the matrix $A = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 0 & -1 \\ -4 & 4 & -5 \end{bmatrix}$. The eigenvalues of this matrix are -1 and -2 . Find all eigenvectors corresponding to the eigenvalue -1 .

8. Below I give two 3×5 matrices A and B together with their reduced row echelon forms:

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 & 4 \\ 1 & 2 & 2 & 4 & 5 \\ 2 & 4 & 1 & 5 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 2 & 3 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 4 & 3 & 1 & 2 & 1 \\ 5 & 4 & 2 & 2 & 1 \\ 7 & 5 & 1 & 4 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 2 & 1 \\ 0 & 1 & 3 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Denote the columns of A by $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5$. Notice that the columns of the matrix B are the columns of the matrix A in reversed order. Therefore $\text{Col } A = \text{Col } B$.

- (a) i. Which basis for $\text{Col } A$ is determined by the given RREF of A ? Call this basis \mathcal{A} .
 ii. Which basis for $\text{Col } B$ is determined by the given RREF of B ? Call this basis \mathcal{B} .

(b) Based on the given RREFs for A and B find $P_{A \leftarrow B}$ and $P_{B \leftarrow A}$.

9. By $\mathbb{R}^{2 \times 2}$ we denote the vector space of all 2×2 matrices. Let A be a matrix in $\mathbb{R}^{2 \times 2}$. Decide which of the following are subspaces and justify your answer.

(a) $\mathcal{H} = \{B \in \mathbb{R}^{2 \times 2} : AB = BA\}$

(b) $\mathcal{H} = \{B \in \mathbb{R}^{2 \times 2} : BA = 0\}$

(c) $\mathcal{H} = \{B \in \mathbb{R}^{2 \times 2} : BB^T = 0\}$

(d) $\mathcal{H} = \{B \in \mathbb{R}^{2 \times 2} : \det(B) = 0\}$

10. Consider the following matrices

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \\ 2 & 1 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 1 \\ 3 & 2 & 3 \\ 4 & 3 & 6 \end{bmatrix}$$

Determine all vectors which belong to both $\text{Col } A$ and $\text{Col } B$.