# MATH 204 Examination November 3, 2019

Name ______________________

GIVE DETAILED EXPLANATIONS FOR YOUR ANSWERS.
On in class exams I assign four problems. Each is worth 25 points.
I try to assign problems from different topics that we covered.
Below are several problems to help you get used to my style of exam questions.

1. Consider $2 \times 3$ matrix
$$A = \begin{bmatrix} -1 & 2 & 3 \\ 2 & -4 & 1 \end{bmatrix}$$
and the linear transformation $T\mathbf{x} = A\mathbf{x}$ which is defined on $\mathbb{R}^3$ and with values in $\mathbb{R}^2$. Determine whether $T$ is injective (one-to-one). Justify your answer. If you claim that this transformation is not injective find two distinct vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ in $\mathbb{R}^3$ such that $T\mathbf{x}_1 = T\mathbf{x}_2$. Determine whether $T$ is surjective (onto). Justify your answer. If you claim that $T$ is surjective then for each $\mathbf{b}$ in $\mathbb{R}^2$ find a vector $\mathbf{x}$ in $\mathbb{R}^3$ such that $T\mathbf{x} = \mathbf{b}$.

2. Consider $3 \times 4$ matrix
$$A = \begin{bmatrix} -1 & 2 & 3 & -6 \\ 2 & -4 & 1 & 5 \\ 1 & -2 & 2 & 1 \end{bmatrix}$$
and the linear transformation $T\mathbf{x} = A\mathbf{x}$ which is defined on $\mathbb{R}^4$ and with values in $\mathbb{R}^3$. Determine whether $T$ is injective (one-to-one). Justify your answer. Determine whether $T$ is surjective (onto). Justify your answer.

3. In this problem $A$ is $n \times n$ matrix. Without using the Invertible Matrix Theorem give explanations for the following implications.
   (a) If the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbb{R}^n$, then there is an $n \times n$ matrix $D$ such that $AD = I$. Explain why.
   (b) If there is an $n \times n$ matrix $D$ such that $AD = I$, then the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbb{R}^n$. Explain why.
   (c) If there is an $n \times n$ matrix $C$ such that $CA = I$, then the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. Explain why.

4. This problem is about invertible matrices. Let $A$ be an $n \times n$ matrix.
   (a) State the definition of an invertible matrix.
   (b) Prove the implication: If $A$ is invertible, then $A$ is row equivalent to $I_n$.
   (c) Prove the implication: If $A$ is row equivalent to $I_n$, then $A$ is invertible.

5. For each matrix below determine whether it is invertible or not. Explain your claim. If a matrix is invertible find its inverse.
$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix},$$

$$D = \begin{bmatrix} 1 & 0 & 1 \\ -1 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix}, \qquad E = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \qquad F = \begin{bmatrix} 1 & -1/2 \\ -2 & 1 \end{bmatrix}.$$

6. Consider the matrix $A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix}$.

   (a) Find $A^{-1}$. Prove that your answer is correct by calculating $AA^{-1}$.

   (b) Use the inverse $A^{-1}$ to find $x_1, x_2, x_3$ such that

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$$

7. Consider the matrices $A = \begin{bmatrix} 1 & -3 \\ -1 & k \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -6 \\ -2 & 3 \end{bmatrix}$

   (a) What values of $k$ (if any) will make $A$ invertible?

   (b) What values of $k$ (if any) will make $AB$ invertible?

   (c) What values of $k$ (if any) will make $AB = BA$?

8. Let $A$ be an unknown $3\times3$ matrix. To $A$ we perform the following row operations:

   - Row 1 gets replaced by $\frac{1}{2}$ times Row 1. ($R_1 \mapsto \frac{1}{2}R_1$)
   - Rows 2 and 3 are swapped ($R_2 \mapsto R_3, R_3 \mapsto R_2$)
   - Row 2 gets replaced by $-3$ Row 2. ($R_2 \mapsto -3R_2$)
   - Row 3 gets replaced by Row 3 minus 6 Row 2. ($R_3 \mapsto R_3 - 6R_2$)

   The resulting matrix $B$ is the identity matrix. What is the matrix $A$?

9. Determine whether it is possible to write the matrix $M = \begin{bmatrix} 1 & 0 & 1 \\ -2 & 1 & -2 \\ 0 & 2 & 1 \end{bmatrix}$ as a product of elementary matrices. If you claim that it is possible to write $M$ as a product of elementary matrices, then find elementary matrices whose product is $M$. If you claim that it is not possible to write $M$ as a product of elementary matrices, then justify your claim.

10. Consider the $3\times5$ matrix

$$A = \begin{bmatrix} -1 & 2 & 3 & 7 & -6 \\ -2 & 4 & 1 & 4 & -7 \\ 1 & -2 & 2 & 3 & 1 \end{bmatrix}$$

   (a) Row reduce the matrix $A$ to the reduced row echelon form.

   (b) Celebrate your correct row reduction by multiplying the $3\times2$ matrix which consists of the pivot columns of $A$ by $2\times5$ matrix which consists of nonzero rows of the reduced row echelon form of $A$.

(c) Find a basis for Nul $A$.

(d) Find a basis for Col $A$.

(e) Express each nonpivot column of $A$ as a linear combination of the basis for Col $A$ that you found.

(f) Find a basis for Row $A$.

(g) Express each row of $A$ as a linear combination of the basis for Row $A$ that you found.