Fall 2019 Math 204 Topics for the third exam

3.1 Introduction to determinants. ➢ Know the definition and the properties (Theorem 1, Theorem 2) of determinants and how to use them to calculate determinants.

3.2 Properties of determinants. ➢ Know how row operations change determinant and how to use this property to calculate determinants.

- ➢ Know that a square matrix is invertible if and only if $\det A \neq 0$.
- ➢ Know the multiplicative property of determinants $\det(AB) = (\det A)(\det B)$ and how to use it to solve problems (Exercises 29, 31).
- ➢ Know the linearity property of the determinant function (page 175) and how to use it to calculate determinants.
- ➢ Know that $\det A^T = \det A$.

3.3 Cramer's rule, volume, and linear transformations. ➢ Know Cramer's rule and how to use it find the unique solution of nonhomogeneous systems with two equations and two unknowns and nonhomogeneous systems with three equations and three unknowns.

- ➢ For a square matrix A with $\det A \neq 0$ know how to use cofactors of of A to write the formula for A^{-1}; see the post on November 13, 2019.
- ➢ Know how to use determinants to calculate areas of parallelograms and triangles and the volumes of parallelepipeds. That is, if A is 2×2 matrix, then the area of the parallelogram determined by the columns of A is $|\det(A)|$. If A is 3×3 matrix, then the volume of the parallelepiped determined by the columns of A is $|\det(A)|$.

4.1 Vector spaces and subspaces. ➢ Know the definition of an abstract vector space; ten axioms: **AE** (addition exist), **AA** (addition is associative), **AC** (addition is commutative), **AZ** (addition has zero), **AO** (addition has opposites), **SE** (scaling exists), **SA** (scaling is "associative"), **SD** (left distributive law), **SD** (right distributive law), **SO** (scaling with one).

- ➢ Know the definition of a subspace and how to use it; three defining properties of a subspace $\mathcal{H}$ are: **SZ** $0 \in \mathcal{U}$, **SA** $u + v \in \mathcal{S}A$ whenever $u, v \in \mathcal{U}$, **SS** $\alpha u \in \mathcal{U}$ whenever $u \in \mathcal{U}$ and $\alpha \in \mathbb{R}$
- ➢ Know the concept of a span
- ➢ Know examples of vector spaces and their subspaces: vector spaces matrices, vector spaces of polynomials and vector spaces of functions

4.2 Null spaces, column spaces and linear transformations. ➢ Know about the null space: the definition, the proof that it is a subspace, how to find a null space of a given matrix, how to write it as a span of vectors, and how to find its basis (this is explained in 4.3)

- ➢ Know about the column space: the definition, how to decide whether a given vector is in the column space of a given matrix, and how to find its basis (this is explained in 4.3)
- ➢ Know the importance of equalities $\operatorname{Nul} A = \{\mathbf{0}\}$ and $\operatorname{Col} A = \mathbb{R}^m$ for a given $m\times n$ matrix A
- ➢ Know the definitions of kernel and range of a linear transformation. Exercises 31, 32, 33

4.3 Linearly independent sets: Bases. ➢ Know the definition of linearly independent vectors and how to prove that given vectors are linearly independent; see the post on November 19, 2019

- ➢ Know the definition of linearly dependent vectors; the characterization of linearly dependent sets in Theorem 4
- ➢ Know the definition of a basis of a vector space and the standard basis for $\mathbb{R}^n$ and $\mathbb{P}_n$
- ➢ For a given $m\times n$ matrix A know how to find bases for $\operatorname{Nul} A$, $\operatorname{Row} A$, and $\operatorname{Col} A$
- ➢ Exercise 23, 24, 34

4.4 Coordinate systems. ➢ For a given vector space $\mathcal{V}$ and its basis $\mathcal{B}$, know the unique representation theorem, the definition of a coordinate mapping, and the meaning of the symbol $[\mathbf{v}]_{\mathcal{B}}$ for a given vector $\mathbf{v}$.

➢ The importance of the matrix

$$P_{\mathcal{B}} = \begin{bmatrix} \mathbf{b}_1 \cdots \mathbf{b}_n \end{bmatrix}$$

for a given basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ for $\mathbb{R}^n$ (this is a special change-of-coordinate matrix, more in Section 4.7)

➢ Theorem 8: Given a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ of a vector space $\mathcal{V}$, the coordinate mapping

$$\mathbf{v} \mapsto [\mathbf{v}]_{\mathcal{B}}$$

is an <u>one-to-one</u> linear transformation from $\mathcal{V}$ <u>onto</u> $\mathbb{R}^n$. In other words, the coordinate mapping $\mathbf{v} \mapsto [\mathbf{v}]_{\mathcal{B}}$ is a linear bijection from $\mathcal{V}$ to $\mathbb{R}^n$.

➢ The coordinate mapping for polynomials, Examples 5 and 6

➢ Exercises 10, 11, 13

4.5 The dimension of a vector space. ➢ Know Theorem 9: Let p and n be positive integers. Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space $\mathcal{V}$. Let $\mathbf{v}_1, \ldots, \mathbf{v}_p$ be vectors in $\mathcal{V}$. If $p > n$, then the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are linearly dependent.

➢ Know Theorem 10: If a vector space $\mathcal{V}$ has a basis of n vectors, then every basis of $\mathcal{V}$ must consist of n vectors.

➢ Kow the definition of a finite dimensional vector space and the definition of the dimension of a finite dimensional vector space

➢ Know Theorem 11 and Theorem 12

➢ For a given $m \times n$ matrix A know how to determine dimensions of $\operatorname{Nul} A$, $\operatorname{Row} A$, and $\operatorname{Col} A$

➢ Exercise 23

4.6 Rank. ➢ The concept of a row space, $\operatorname{Row} A$, of a given matrix A

➢ Know Theorem 13: If two matrices A and B are row equivalent, then their row spaces are the same. If B is in row echelon form, then the nonzero rows of B form a basis for the row space of A (which is the same as the row space of B).

➢ Know the definition of the rank of a matrix

➢ Know that the nullity of a matrix A is the dimension of $\operatorname{Nul} A$

➢ Kow that the Rank Theorem in the book is more often called the Rank-Nullity theorem. This theorem has three important claims.

➢ Know **The Rank-Nullity Theorem**: **(1)** The dimensions of the column space and the row space of an $m \times n$ matrix A are equal. **(2)** This common dimension, the rank of A, also equals the number of pivot positions in A. **(3)** The rank of A and the dimension of $\operatorname{Nul} A$ add up to the number of columns of A. That is

$$\operatorname{rank} A + \dim \operatorname{Nul} A = n.$$

➢ Know: Both $\operatorname{Nul} A$ and $\operatorname{Row} A$ are subspaces of $\mathbb{R}^n$. The only vector which is in both $\operatorname{Nul} A$ and $\operatorname{Row} A$ is the zero vector. A union of a basis for $\operatorname{Nul} A$ and a basis for $\operatorname{Row} A$ is a basis for $\mathbb{R}^n$.

➢ Four fundamental subspaces determined by A and relationships among their dimensions.

➢ Exercises 27-30 among others

4.7 Change of bases (in fact: Change of coordinates). ➢ Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_m\}$ and $\mathcal{C} = \{\mathbf{c}_1, \ldots, \mathbf{c}_m\}$ be two bases for a vector space $\mathcal{V}$. Know that the matrix M with the property $[\mathbf{v}]_{\mathcal{C}} = M[\mathbf{v}]_{\mathcal{B}}$ is called the **change-of-coordinate matrix from $\mathcal{B}$ to $\mathcal{C}$**. It is denoted $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$ and it is calculated as

$$\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = \begin{bmatrix} [\mathbf{b}_1]_{\mathcal{C}} & \cdots & [\mathbf{b}_m]_{\mathcal{C}} \end{bmatrix}$$

- Know that $\left(\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\right)^{-1} = \underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$
- If the vector space $\mathcal{V}$ is a subspace of $\mathbb{R}^n$, then to determine the vector $[\mathbf{b}_1]_{\mathcal{C}}$ we have to solve the vector equation

$$x_1\mathbf{c}_1 + \cdots + x_m\mathbf{c}_m = \mathbf{b}_1.$$

To solve this vector equation we would row reduce the augmented matrix

$$\left[\begin{array}{ccc|c} \mathbf{c}_1 & \cdots & \mathbf{c}_m & \mathbf{b}_1 \end{array}\right].$$

Since the given vector equation has a unique solution, the row reduction will give that solution in the last column, that is, in the column after $|$. To get the coordinate vectors $[\mathbf{b}_2]_{\mathcal{C}}, \ldots, [\mathbf{b}_m]_{\mathcal{C}}$ for other vectors in $\mathcal{B}$ we can row reduce

$$\left[\begin{array}{ccc|ccc} \mathbf{c}_1 & \cdots & \mathbf{c}_m & \mathbf{b}_1 & \cdots & \mathbf{b}_m \end{array}\right].$$

The row reduction will result in

$$\left[\begin{array}{ccc|ccc} \mathbf{c}_1 & \cdots & \mathbf{c}_m & \mathbf{b}_1 & \cdots & \mathbf{b}_m \end{array}\right] \sim \cdots \sim \left[\begin{array}{c|c} I_m & \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} \\ 0 & 0 \end{array}\right].$$

See the post on November 21, 2019.

- In the above row reduction I assumed that $m < n$. If $m = n$, then the zeros in the RREF are not present. We have

$$\left[\begin{array}{ccc|ccc} \mathbf{c}_1 & \cdots & \mathbf{c}_n & \mathbf{b}_1 & \cdots & \mathbf{b}_n \end{array}\right] \sim \cdots \sim \left[\begin{array}{c|c} I_n & \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} \end{array}\right].$$

- Know that there is a special basis of $\mathbb{R}^n$, called the standard basis, which consists of the columns of the identify matrix I_n. These vectors are denoted by $\mathbf{e}_1, \ldots, \mathbf{e}_n$ and the basis consisting of these vectors is denoted by $\mathcal{E}$. The above considerations show that

$$\underset{\mathcal{E}\leftarrow\mathcal{B}}{P} = \left[\begin{array}{ccc} \mathbf{b}_1 & \cdots & \mathbf{b}_n \end{array}\right]$$

- Know that in the vector space of polynomials $\mathbb{P}_n$ we have a special basis which consists of monomials $1, x, x^2, \ldots, x^n$. Denote this basis by $\mathcal{M}$

$$\mathcal{M} = \{1, x, x^2, \ldots, x^n\}$$

If we have two bases $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_m\}$ and $\mathcal{C} = \{\mathbf{c}_1, \ldots, \mathbf{c}_m\}$ for a subspace $\mathcal{V}$ of $\mathbb{P}_n$, then to get $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ we row reduce the matrix

$$\left[\begin{array}{ccc|ccc} [\mathbf{c}_1]_{\mathcal{M}} & \cdots & [\mathbf{c}_m]_{\mathcal{M}} & [\mathbf{b}_1]_{\mathcal{M}} & \cdots & [\mathbf{b}_m]_{\mathcal{M}} \end{array}\right] \sim \cdots \sim \left[\begin{array}{c|c} I_m & \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} \\ 0 & 0 \end{array}\right].$$

- Exercises 4 - 10, 13, 14.