Fall 2019 Math 204 Topics for the Final Exam

1.1 Linear systems. ➢ Know the definition of a linear equation and the definition of a linear system;

- ➢ Know what is the coefficient matrix and the augmented matrix of a linear system;
- ➢ Know what are equivalent linear systems and what are three basic operations which transform a linear system into an equivalent simpler system;
- ➢ Know the existence and uniqueness questions for linear system and how to answer them.

1.2 Row reduction, row echelon form (REF) and reduced row echelon form (RREF). ➢ Know the definitions of REF and RREF of a matrix and how to use the Row Reduction Algorithm to transform a matrix to RREF.

- ➢ Know the concepts of a pivot position and a pivot column in a matrix and the connection with the basic and free variables of a system.
- ➢ Know how to use row reduction to find the general solution of a linear system and how to write this solution in parametric vector form.
- ➢ Know the Existence and Uniqueness Theorem (Theorem 2 page 21).

1.3 Vector equations. ➢ Know how to write a linear system as one vector equation.

- ➢ It is essential to know how to verify whether row reduction has been performed correctly: **The linear relationships among vectors in the given matrix and its RREF are the same.**
- ➢ Know algebraic operations in the vector space $\mathbb{R}^n$, their geometric illustrations in $\mathbb{R}^2$ and $\mathbb{R}^3$.
- ➢ Know the concepts of a linear combination of vectors and a span of vectors and the geometric interpretation of a span in $\mathbb{R}^2$ and $\mathbb{R}^3$.

1.4 The matrix equation $A\mathbf{x} = \mathbf{b}$. ➢ Know the definition of matrix-vector product and its basic properties: $A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v}$, $A(c\mathbf{u}) = cA\mathbf{u}$ (Theorem 5).

- ➢ Know the matrix equation, the vector equation and the linear system which have the same solution set (Theorem 3).
- ➢ Know four equivalent ways of saying: For every $\mathbf{b} \in \mathbb{R}^m$ the equation $A\mathbf{x} = \mathbf{b}$ has a solution (Theorem 4).

1.5 Solutions sets of linear systems. ➢ Know the geometric illustration of the expression $\mathbf{p} + t\mathbf{v}$ where t is an arbitrary scalar and $\mathbf{p}$ and $\mathbf{v}$ are fixed vectors in $\mathbb{R}^n$.

- ➢ Know the geometric illustration of the expression $\mathbf{p} + s\mathbf{u} + t\mathbf{v}$ where t and s are arbitrary scalars and $\mathbf{p}$, $\mathbf{u}$ and $\mathbf{v}$ are fixed vectors in $\mathbb{R}^n$.
- ➢ Know how to write a solution of a linear system in parametric vector form.
- ➢ Know the relationship between the solution sets of a nonhomogeneous equation $A\mathbf{x} = \mathbf{b}$ and the corresponding homogeneous equation $A\mathbf{x} = \mathbf{0}$ (Theorem 6).

1.7 Linear independence. ➢ Know the definitions of linear independence and linear dependence and how to implement them to decide whether given vectors are linearly dependent or independent.

- ➢ Know the meaning of linear independence/dependence in the case of one vector and two vectors.
- ➢ Know two simple sufficient conditions for the linear dependence (Theorems 8 and 9).
- ➢ Know a characterization of linearly dependent sets (Theorems 7).

1.8 Linear transformations. ➢ Know that in this context the words transformation, mapping and function are synonyms.

- ➢ Know the definition of a **linear transformation**. Let n and m be positive integers. A transformation T defined on $\mathbb{R}^n$ and with the values in $\mathbb{R}^m$ is **linear** if the following two conditions are satisfied: $T(\mathbf{x}+\mathbf{y}) = T\mathbf{x} + T\mathbf{y}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $T(c\mathbf{x}) = cT\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$ and all $c \in \mathbb{R}$.

- ➢ Know how to associate pictures to formulas and formulas to pictures.

1.9 Matrix of a linear transformations. ➢ Know the most important theorem on the **standard matrix for a linear transformation**: If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then there exists a unique $m \times n$ matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$. In fact, the jth column of A is $T(\mathbf{e}_j)$, where $\mathbf{e}_j$ is the jth column of the identity matrix.

- ➢ Know how to use the above theorem to get the standard matrices of transformations in Tables 1, 2, 3, 4 in Section 1.9. Also Exercises 1-22 all use this idea.
- ➢ In particular, know how to use the above theorem to get the standard matrix for a **rotation** about the origin in $\mathbb{R}^2$ and a **reflection** through a line passing through the origin in $\mathbb{R}^2$ (which are linear transformations of $\mathbb{R}^2$).
- ➢ Definitions of **injection** (**one-to-one**) and **surjection** (**onto**) for linear transformations and the characterizations in Theorems 11 and 12. Exercises 23-30.

2.1 Matrix operations. ➢ Know how to add matrices and multiply matrices with a scalar and properties of these operations.

- ➢ Know how to multiply two matrices (the definition and row-column rule for the computation).
- ➢ Know properties of matrix multiplication.
- ➢ Know the content of the post on October 22: For a given matrix A, find the Reduced Row Echelon Form (RREF) of A, then form the matrix which consists of the pivot columns of A and the matrix which consists of the nonzero rows of the RREF of A. What is the product of these two matrices?
- ➢ Know the definition of the transpose of a matrix and its properties.

2.2 The inverse of a matrix. ➢ Know the definition of an invertible matrix and the definition of an inverse of a matrix.

- ➢ Know the easy inverses: 2×2 matrices, elementary matrices, product of invertible matrices.
- ➢ Know the algorithm for finding A^{-1} and its connection with elementary matrices, see the post on October 25.
- ➢ Based on the post on October 25 be able to write an invertible matrix as a product of elementary matrices.
- ➢ Know how to use inverse to solve the equation $A\mathbf{x} = \mathbf{b}$.
- ➢ Know Theorem 7 and its proof.

2.3 Characterization of invertible matrices. ➢ Know the statement and the proof of the invertible matrix theorem. (See examples of the proofs that we did in class.)

2.8 Subspaces of $\mathbb{R}^n$. ➢ Know the definition of a subspace of $\mathbb{R}^n$.

- ➢ Know that Example 3 gives the most important example of a subspace.
- ➢ Know the definition of a basis of a subspace of $\mathbb{R}^n$.
- ➢ Know the definition of $\operatorname{Col} A$ (for a given $n\times m$ matrix A) and how to find a basis for $\operatorname{Col} A$.
- ➢ Know the definition of $\operatorname{Nul} A$ (for a given $n\times m$ matrix A) and how to find a basis for $\operatorname{Nul} A$.
- ➢ Know the definition of $\operatorname{Row} A$ (for a given $n\times m$ matrix A) and how to find a basis for $\operatorname{Row} A$. (see the post of November 1)
- ➢ Know the post of October 31.

2.9 Dimension and rank. ➢ Know the definition of the **dimension** of a subspace of $\mathbb{R}^n$.

- ➢ Know the definition of the rank of an $m \times n$ matrix A.
- ➢ Know that for an $m \times n$ matrix A the dimension of the column space of A **equals** the dimension of the row space of A.

- ➢ Know the rank theorem for an $m \times n$ matrix A: The dimension of the column space plus the dimension of the null space of A equals the number of columns in A.

3.1 Introduction to determinants. ➢ Know the definition and the properties (Theorem 1, Theorem 2) of determinants and how to use them to calculate determinants.

3.2 Properties of determinants. ➢ Know how row operations change determinant and how to use this property to calculate determinants.

- ➢ Know that a square matrix is invertible if and only if $\det A \neq 0$.
- ➢ Know the multiplicative property of determinants $\det(AB) = (\det A)(\det B)$ and how to use it to solve problems (Exercises 29, 31).
- ➢ Know the linearity property of the determinant function (page 175) and how to use it to calculate determinants.
- ➢ Know that $\det A^T = \det A$.

3.3 Cramer's rule, volume, and linear transformations. ➢ Know Cramer's rule and how to use it find the unique solution of nonhomogeneous systems with two equations and two unknowns and nonhomogeneous systems with three equations and three unknowns.

- ➢ For a square matrix A with $\det A \neq 0$ know how to use cofactors of A to write the formula for A^{-1}; see the post on November 13, 2019.
- ➢ Know how to use determinants to calculate areas of parallelograms and triangles and the volumes of parallelepipeds. That is, if A is 2×2 matrix, then the area of the parallelogram determined by the columns of A is $|\det(A)|$. If A is 3×3 matrix, then the volume of the parallelepiped determined by the columns of A is $|\det(A)|$.

4.1 Vector spaces and subspaces. ➢ Know the definition of an abstract vector space; ten axioms: **AE** (addition exist), **AA** (addition is associative), **AC** (addition is commutative), **AZ** (addition has zero), **AO** (addition has opposites), **SE** (scaling exists), **SA** (scaling is "associative"), **SD** (left distributive law), **SD** (right distributive law), **SO** (scaling with one).

- ➢ Know the definition of a subspace and how to use it; three defining properties of a subspace $\mathcal{H}$ are: **SZ** $0 \in \mathcal{U}$, **SA** $u + v \in \mathcal{S}A$ whenever $u, v \in \mathcal{U}$, **SS** $\alpha u \in \mathcal{U}$ whenever $u \in \mathcal{U}$ and $\alpha \in \mathbb{R}$
- ➢ Know the concept of a span
- ➢ Know examples of vector spaces and their subspaces: vector spaces matrices, vector spaces of polynomials and vector spaces of functions

4.2 Null spaces, column spaces and linear transformations. ➢ Know about the null space: the definition, the proof that it is a subspace, how to find a null space of a given matrix, how to write it as a span of vectors, and how to find its basis (this is explained in 4.3)

- ➢ Know about the column space: the definition, how to decide whether a given vector is in the column space of a given matrix, and how to find its basis (this is explained in 4.3)
- ➢ Know the importance of equalities $\operatorname{Nul} A = \{\mathbf{0}\}$ and $\operatorname{Col} A = \mathbb{R}^m$ for a given $m\times n$ matrix A
- ➢ Know the definitions of kernel and range of a linear transformation. Exercises 31, 32, 33

4.3 Linearly independent sets: Bases. ➢ Know the definition of linearly independent vectors and how to prove that given vectors are linearly independent; see the post on November 19, 2019

- ➢ Know the definition of linearly dependent vectors; the characterization of linearly dependent sets in Theorem 4
- ➢ Know the definition of a basis of a vector space and the standard basis for $\mathbb{R}^n$ and $\mathbb{P}_n$
- ➢ For a given $m\times n$ matrix A know how to find bases for $\operatorname{Nul} A$, $\operatorname{Row} A$, and $\operatorname{Col} A$

- ➢ Exercise 23, 24, 34

4.4 Coordinate systems. ➢ For a given vector space $\mathcal{V}$ and its basis $\mathcal{B}$, know the unique representation theorem, the definition of a coordinate mapping, and the meaning of the symbol $[\mathbf{v}]_{\mathcal{B}}$ for a given vector $\mathbf{v}$.

- ➢ The importance of the matrix
$$P_{\mathcal{B}} = \begin{bmatrix} \mathbf{b}_1 & \cdots & \mathbf{b}_n \end{bmatrix}$$
for a given basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ for $\mathbb{R}^n$ (this is a special change-of-coordinate matrix, more in Section 4.7)
- ➢ Theorem 8: Given a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ of a vector space $\mathcal{V}$, the coordinate mapping
$$\mathbf{v} \mapsto [\mathbf{v}]_{\mathcal{B}}$$
is an one-to-one linear transformation from $\mathcal{V}$ onto $\mathbb{R}^n$. In other words, the coordinate mapping $\mathbf{v} \mapsto [\mathbf{v}]_{\mathcal{B}}$ is a linear bijection from $\mathcal{V}$ to $\mathbb{R}^n$.
- ➢ The coordinate mapping for polynomials, Examples 5 and 6
- ➢ Exercises 10, 11, 13

4.5 The dimension of a vector space. ➢ Know Theorem 9: Let p and n be positive integers. Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space $\mathcal{V}$. Let $\mathbf{v}_1, \ldots, \mathbf{v}_p$ be vectors in $\mathcal{V}$. If $p > n$, then the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are linearly dependent.

- ➢ Know Theorem 10: If a vector space $\mathcal{V}$ has a basis of n vectors, then every basis of $\mathcal{V}$ must consist of n vectors.
- ➢ Know the definition of a finite dimensional vector space and the definition of the dimension of a finite dimensional vector space
- ➢ Know Theorem 11 and Theorem 12
- ➢ For a given $m \times n$ matrix A know how to determine dimensions of $\operatorname{Nul} A$, $\operatorname{Row} A$, and $\operatorname{Col} A$
- ➢ Exercise 23

4.6 Rank. ➢ The concept of a row space, $\operatorname{Row} A$, of a given matrix A

- ➢ Know Theorem 13: If two matrices A and B are row equivalent, then their row spaces are the same. If B is in row echelon form, then the nonzero rows of B form a basis for the row space of A (which is the same as the row space of B).
- ➢ Know the definition of the rank of a matrix
- ➢ Know that the nullity of a matrix A is the dimension of $\operatorname{Nul} A$
- ➢ Know that the Rank Theorem in the book is more often called the Rank-Nullity theorem. This theorem has three important claims.
- ➢ Know **The Rank-Nullity Theorem**: **(1)** The dimensions of the column space and the row space of an $m \times n$ matrix A are equal. **(2)** This common dimension, the rank of A, also equals the number of pivot positions in A. **(3)** The rank of A and the dimension of $\operatorname{Nul} A$ add up to the number of columns of A. That is
$$\operatorname{rank} A + \dim \operatorname{Nul} A = n.$$
- ➢ Know: Both $\operatorname{Nul} A$ and $\operatorname{Row} A$ are subspaces of $\mathbb{R}^n$. The only vector which is in both $\operatorname{Nul} A$ and $\operatorname{Row} A$ is the zero vector. A union of a basis for $\operatorname{Nul} A$ and a basis for $\operatorname{Row} A$ is a basis for $\mathbb{R}^n$.
- ➢ Four fundamental subspaces determined by A and relationships among their dimensions.
- ➢ Exercises 27-30 among others

4.7 Change of bases (in fact: Change of coordinates). ➢ Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_m\}$ and $\mathcal{C} = \{\mathbf{c}_1, \ldots, \mathbf{c}_m\}$ be two bases for a vector space $\mathcal{V}$. Know that the matrix M with the property $[\mathbf{v}]_\mathcal{C} = M[\mathbf{v}]_\mathcal{B}$ is called the **change-of-coordinate matrix from $\mathcal{B}$ to $\mathcal{C}$**. It is denoted $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ and it is calculated as

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} [\mathbf{b}_1]_\mathcal{C} & \cdots & [\mathbf{b}_m]_\mathcal{C} \end{bmatrix}$$

➢ Know that $\left(\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\right)^{-1} = \underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$

➢ If the vector space $\mathcal{V}$ is a subspace of $\mathbb{R}^n$, then to determine the vector $[\mathbf{b}_1]_\mathcal{C}$ we have to solve the vector equation

$$x_1\mathbf{c}_1 + \cdots + x_m\mathbf{c}_m = \mathbf{b}_1.$$

To solve this vector equation we would row reduce the augmented matrix

$$\left[\begin{array}{ccc|c} \mathbf{c}_1 & \cdots & \mathbf{c}_m & \mathbf{b}_1 \end{array}\right].$$

Since the given vector equation has a unique solution, the row reduction will give that solution in the last column, that is, in the column after $|$. To get the coordinate vectors $[\mathbf{b}_2]_\mathcal{C}, \ldots, [\mathbf{b}_m]_\mathcal{C}$ for other vectors in $\mathcal{B}$ we can row reduce

$$\left[\begin{array}{ccc|ccc} \mathbf{c}_1 & \cdots & \mathbf{c}_m & \mathbf{b}_1 & \cdots & \mathbf{b}_m \end{array}\right].$$

The row reduction will result in

$$\left[\begin{array}{ccc|ccc} \mathbf{c}_1 & \cdots & \mathbf{c}_m & \mathbf{b}_1 & \cdots & \mathbf{b}_m \end{array}\right] \sim \cdots \sim \left[\begin{array}{c|c} I_m & \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} \\ 0 & 0 \end{array}\right].$$

See the post on November 21, 2019.

➢ In the above row reduction I assumed that $m < n$. If $m = n$, then the zeros in the RREF are not present. We have

$$\left[\begin{array}{ccc|ccc} \mathbf{c}_1 & \cdots & \mathbf{c}_n & \mathbf{b}_1 & \cdots & \mathbf{b}_n \end{array}\right] \sim \cdots \sim \left[\begin{array}{c|c} I_n & \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} \end{array}\right].$$

➢ Know that there is a special basis of $\mathbb{R}^n$, called the standard basis, which consists of the columns of the identify matrix I_n. These vectors are denoted by $\mathbf{e}_1, \ldots, \mathbf{e}_n$ and the basis consisting of these vectors is denoted by $\mathcal{E}$. The above considerations show that

$$\underset{\mathcal{E}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} \mathbf{b}_1 & \cdots & \mathbf{b}_n \end{bmatrix}$$

➢ Know that in the vector space of polynomials $\mathbb{P}_n$ the standard basis consists of monomials $1, x, x^2, \ldots, x^n$. Denote this basis by $\mathcal{M}$

$$\mathcal{M} = \{1, x, x^2, \ldots, x^n\}$$

If we have two bases $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_m\}$ and $\mathcal{C} = \{\mathbf{c}_1, \ldots, \mathbf{c}_m\}$ for a subspace $\mathcal{V}$ of $\mathbb{P}_n$, then to get $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ we row reduce the matrix

$$\left[\begin{array}{ccc|ccc} [\mathbf{c}_1]_\mathcal{M} & \cdots & [\mathbf{c}_m]_\mathcal{M} & [\mathbf{b}_1]_\mathcal{M} & \cdots & [\mathbf{b}_m]_\mathcal{M} \end{array}\right] \sim \cdots \sim \left[\begin{array}{c|c} I_m & \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} \\ 0 & 0 \end{array}\right].$$

➢ Exercises 4 - 10, 13, 14.

5.1 Eigenvectors and eigenvalues. ➢ Know the definition of an eigenvector and eigenvalue. It is a little tricky. Pay attention.

➢ Know the definition of an eigenspace and how to find an eigenspace corresponding to a given eigenvalue.

- Know that the eigenvalues of a triangular matrix are the entries on its main diagonal.
- **Theorem.** Eigenvectors corresponding to distinct eigenvalues are linearly independent. (Or, in more formal mathematical language: Let A be an $n \times n$ matrix, let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m \in \mathbb{R}^n$ and $\lambda_1, \lambda_2, \ldots, \lambda_m \in \mathbb{R}$. If $A\vec{v}_k = \lambda_k \vec{v}_k$, $\vec{v}_k \neq \vec{0}$ and $\lambda_j \neq \lambda_k$ for all $j, k = 1, 2, \ldots, m$, then $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ are linearly independent.
- Know a proof of the above theorem for $m = 2$ and $m = 3$ vectors.

5.2 The characteristic equation.
- Know that λ is an eigenvalue of an $n \times n$ matrix A if and only if $\det(A - \lambda I) = 0$
- Know how to calculate $\det(A - \lambda I)$ (this is the characteristic polynomial) for 2×2 matrices and 3×3 matrices, how to find eigenvalues and corresponding eigenvectors. Exercises 1-8, but do more and find eigenvectors as well.

5.3 Diagonalization.
- **Theorem.** (The diagonalization theorem) An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.
- Know how to decide whether a given 2×2 and 3×3 matrix A, is diagonalizable or not; if it is diagonalizable, know how to find an invertible matrix P and diagonal matrix D such that $A = PDP^{-1}$.
- Know how to decide whether a triangular matrix is diagonalizable or not. Consider the matrix A in Exercise 18 in Section 5.2 and find h such that the matrix A is diagonalizable.