MATH 204 Examination November 9, 2024 Name ____________________

GIVE DETAILED EXPLANATIONS FOR YOUR ANSWERS.
On in class exams I assign four problems. Each is worth 25 points.
I try to assign problems from different topics that we covered.
Below are several problems to help you get used to my style of exam questions.

1. Consider 2×3 matrix
$$A = \begin{bmatrix} -1 & 2 & 3 \\ 2 & -4 & 1 \end{bmatrix}$$
and the linear transformation $T\mathbf{x} = A\mathbf{x}$ which is defined on $\mathbb{R}^3$ and with values in $\mathbb{R}^2$. Determine whether T is injective (one-to-one). Justify your answer. If you claim that this transformation is not injective find two distinct vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ in $\mathbb{R}^3$ such that $T\mathbf{x}_1 = T\mathbf{x}_2$. Determine whether T is surjective (onto). Justify your answer. If you claim that T is surjective then for each $\mathbf{b}$ in $\mathbb{R}^2$ find a vector $\mathbf{x}$ in $\mathbb{R}^3$ such that $T\mathbf{x} = \mathbf{b}$.

2. Consider 3×4 matrix
$$A = \begin{bmatrix} -1 & 2 & 3 & -6 \\ 2 & -4 & 1 & 5 \\ 1 & -2 & 2 & 1 \end{bmatrix}$$
and the linear transformation $T\mathbf{x} = A\mathbf{x}$ which is defined on $\mathbb{R}^4$ and with values in $\mathbb{R}^3$. Determine whether T is injective (one-to-one). Justify your answer. Determine whether T is surjective (onto). Justify your answer.

3. (a) Define the set
$$\mathcal{S} = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : |y - x| = |y - z| \right\}.$$
Here $|\alpha|$ denotes the absolute value of a real number α. Is $\mathcal{S}$ a subspace of $\mathbb{R}^3$? Why or why not?

   (b) Consider now the set
$$\mathcal{H} = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : z = x \right\}.$$
Note that $\mathcal{H}$ is a *subset* of $\mathcal{S}$. Is $\mathcal{H}$ a subspace of $\mathbb{R}^3$? Why or why not?

4. In this problem we assume that n is an integer such that $n > 1$. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$; that is $\mathbf{u}$ and $\mathbf{v}$ are $n\times 1$ matrices. Consider the following expressions
$$\mathbf{u}^\top \mathbf{v}, \qquad \mathbf{v}^\top \mathbf{u}, \qquad \mathbf{v}\,\mathbf{u}^\top, \qquad \mathbf{u}\,\mathbf{v}^\top.$$
Here, the symbol $^\top$ denotes the transpose of a matrix. Give detailed answers to the following questions:

   (a) Is it possible that $\mathbf{u}^\top \mathbf{v} = \mathbf{v}^\top \mathbf{u}$?

   (b) Is it possible that $\mathbf{u}^\top \mathbf{v} = \mathbf{v}\,\mathbf{u}^\top$?

   (c) Is it possible that $\mathbf{u}\,\mathbf{v}^\top = \mathbf{v}\,\mathbf{u}^\top$?

(d) Does the expression $(\mathbf{u}\,\mathbf{v}^\top)\mathbf{v}$ make sense? If this expression makes sense, which kind of matrix is it?

5. Let $n \in \mathbb{N}$. In this problem A is an $n \times n$ matrix. Without using the Invertible Matrix Theorem give explanations for the following implications.

   (a) If the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbb{R}^n$, then there is an $n \times n$ matrix D such that $AD = I$. Explain why.

   (b) If there is an $n \times n$ matrix D such that $AD = I$, then the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbb{R}^n$. Explain why.

   (c) If there is an $n \times n$ matrix C such that $CA = I$, then the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. Explain why.

6. This problem is about invertible matrices. Let A be an $n \times n$ matrix.

   (a) State the definition of an invertible matrix.

   (b) Prove the implication: If A is invertible, then A is row equivalent to I_n.

   (c) Prove the implication: If A is row equivalent to I_n, then A is invertible.

7. For each matrix below determine whether it is invertible or not. Explain your claim. If a matrix is invertible find its inverse.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix},$$

$$D = \begin{bmatrix} 1 & 0 & 1 \\ -1 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix}, \qquad E = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \qquad F = \begin{bmatrix} 1 & -1/2 \\ -2 & 1 \end{bmatrix}.$$

8. Consider the matrix $A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix}$.

   (a) Find A^{-1}. Prove that your answer is correct by calculating AA^{-1}.

   (b) Use the inverse A^{-1} to find x_1, x_2, x_3 such that

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$$

9. Determine h such that

$$\begin{vmatrix} 1 & 2 & 3 & 0 \\ 0 & 2 & 1 & h \\ 2 & -2 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{vmatrix} = 0.$$

10. Let a, b, c, d be real numbers. Calculate the following three determinants:

$$\begin{vmatrix} a & 0 & b \\ 0 & 1 & 0 \\ c & 0 & d \end{vmatrix}, \qquad \begin{vmatrix} a & 0 & 0 & 0 & b \\ 0 & a & 0 & b & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & c & 0 & d & 0 \\ c & 0 & 0 & 0 & d \end{vmatrix}, \qquad \begin{vmatrix} a & 0 & 0 & 0 & 0 & 0 & b \\ 0 & a & 0 & 0 & 0 & b & 0 \\ 0 & 0 & a & 0 & b & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & c & 0 & d & 0 & 0 \\ 0 & c & 0 & 0 & 0 & d & 0 \\ c & 0 & 0 & 0 & 0 & 0 & d \end{vmatrix}.$$

11. Consider the matrices $A = \begin{bmatrix} 1 & -3 \\ -1 & k \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -6 \\ -2 & 3 \end{bmatrix}$

    (a) What values of k (if any) will make A invertible?

    (b) What values of k (if any) will make AB invertible?

    (c) What values of k (if any) will make $AB = BA$?

12. For which value of λ the matrix

$$\begin{bmatrix} -x - \lambda & 3 \\ -6 & 5 - \lambda \end{bmatrix}$$

    is invertible?

13. Let A be an unknown square matrix. To A we perform the following row operations:

    - Row 1 gets replaced by $\frac{1}{2}$ times Row 1. ($R_1 \mapsto \frac{1}{2}R_1$)
    - Rows 2 and 3 are swapped ($R_2 \mapsto R_3, R_3 \mapsto R_2$)
    - Row 2 gets replaced by -3 Row 2. ($R_2 \mapsto -3R_2$)
    - Row 3 gets replaced by Row 3 minus 6 Row 2. ($R_3 \mapsto R_3 - 6R_2$)

    The resulting matrix B has determinant $\det B = 4$. What is the determinant of the unknown matrix A.

14. Let A be an unknown 3×3 matrix. To A we perform the following row operations:

    - Row 1 gets replaced by $\frac{1}{2}$ times Row 1. ($R_1 \mapsto \frac{1}{2}R_1$)
    - Rows 2 and 3 are swapped ($R_2 \mapsto R_3, R_3 \mapsto R_2$)
    - Row 2 gets replaced by -3 Row 2. ($R_2 \mapsto -3R_2$)
    - Row 3 gets replaced by Row 3 minus 6 Row 2. ($R_3 \mapsto R_3 - 6R_2$)

    The resulting matrix B is the identity matrix. What is the matrix A?

15. Prove that

$$\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = (y - x)(z - x)(z - y).$$

16. Determine whether it is possible to write the matrix $M = \begin{bmatrix} 1 & 0 & 1 \\ -2 & 1 & -2 \\ 0 & 2 & 1 \end{bmatrix}$ as a product of elementary matrices. If you claim that it is possible to write M as a product of elementary matrices, then find elementary matrices whose product is M. If you claim that it is not possible to write M as a product of elementary matrices, then justify your claim.

17. Calculate the determinant $\begin{vmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 5 & 0 & 6 & 7 \\ 0 & 0 & 8 & 9 \end{vmatrix}$.

18. Determine h such that

$$\begin{vmatrix} 1 & 2 & 3 & 0 \\ 0 & 2 & 1 & h \\ 2 & -2 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{vmatrix} = 0.$$

19. (This problem has too many items for all of them to be on an exam.) Consider the 3×5 matrix

$$A = \begin{bmatrix} -1 & 2 & 3 & 7 & -6 \\ -2 & 4 & 1 & 4 & -7 \\ 1 & -2 & 2 & 3 & 1 \end{bmatrix}$$

   (a) Row reduce the matrix A to the reduced row echelon form.

   (b) Celebrate your correct row reduction by multiplying the 3×2 matrix which consists of the pivot columns of A by 2×5 matrix which consists of nonzero rows of the reduced row echelon form of A.

   (c) Find a basis for $\operatorname{Nul} A$.

   (d) Find a basis for $\operatorname{Col} A$.

   (e) Express each nonpivot column of A as a linear combination of the basis for $\operatorname{Col} A$ that you found.

   (f) Find a basis for $\operatorname{Row} A$.

   (g) Express each row of A as a linear combination of the basis for $\operatorname{Row} A$ that you found.

20. The matrix A and its reduced row echelon form are given below.

$$A = \begin{bmatrix} 1 & 1 & 3 & 2 & 1 & 1 \\ 1 & 2 & 5 & 3 & 0 & 3 \\ 2 & 3 & 8 & 5 & 1 & 4 \\ 2 & 2 & 6 & 4 & 3 & 1 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   (a) Find the rank of A and the dimension of the null space of A.

   (b) Find the rank of $A^\top$ and the dimension of the null space of $A^\top$.

   (c) Denote the columns of A by $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, $\mathbf{a}_4$, $\mathbf{a}_5$, $\mathbf{a}_6$. Based on the given RREF of A, find a basis for the column space of A. Denote this basis by $\mathcal{C}$. Calculate the vector $[\mathbf{a}_6]_{\mathcal{C}}$.

   (d) Express the first row of A as a linear combination of the nonzeor rows of the RREF of A.

(e) Find a basis for the null space of A.

(f) Prove that the set

$$\mathcal{D} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is a basis for the column space of A.

(g) Calculate the change of coordinates matrix $\underset{\mathcal{D} \leftarrow \mathcal{C}}{P}$

21. Given two bases $\mathcal{A}$ and $\mathcal{B}$ calculate the change of coordinates matrices $\underset{\mathcal{A} \leftarrow \mathcal{B}}{P}$ and $\underset{\mathcal{B} \leftarrow \mathcal{A}}{P}$. There are several examples on the class website. One is with a picture, one is with given vectors.

22. Let

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

and let

$$\mathcal{H} = \operatorname{Span} \mathcal{B}.$$

(a) Prove that $\mathcal{B}$ is a basis for $\mathcal{H}$.

(b) Find the basis $\mathcal{C}$ of $\mathcal{H}$ such that

$$\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}.$$