MATH 209

Assignment 1
May 18, 2012

Name ____________________

Problem 1. Recall that the Fibonacci sequence is defined as

$$f_0 = 0, \quad f_1 = 1, \qquad f_{n+1} = f_n + f_{n-1}, \quad n \in \mathbb{Z}_+.$$

Recall that the Golden ratio ϕ is a positive solution of the equation $\phi^2 = \phi + 1$.

(a) Prove that $\psi^2 = \psi + 1$ if and only if for all $n \in \mathbb{Z}_+$ we have $\psi^n = f_{n-1} + \psi f_n$.

(b) Prove that for all $n \in \mathbb{Z}_+$ we have $f_n = \dfrac{\phi^n - (-\phi)^{-n}}{\phi + \phi^{-1}}$. (You can use (a) here. Notice that in (a) you have proved two if and only if statements.)

Problem 2. Consider the recursion

$$p_0 = 1, \qquad p_{n+1} = \sum_{k=0}^{n} p_k, \quad n \in \mathbb{N}.$$

Discover a simple formula for p_n (a formula that will express p_n as a function of n only) and prove it. A formula that works for all $n \in \mathbb{Z}_+$ is sufficient.

Problem 3. Let $n \in \mathbb{Z}_+$. Denote by g_n the number of bit strings of length n with no consecutive 0s.

(a) Calculate g_1, g_2, g_3 and g_4. You can proceed with $g_5, g_6, \ldots$. The pattern should be clear.

(b) Based on (a) it should be clear what is the recursive formula for the numbers $g_n, n \in \mathbb{Z}_+$. State this formula clearly.

(c) Prove the formula in (b).

Problem 4. Let $p \to q$ be an implication. We will call this implication "the original implication". State clearly what is the **contrapositive**, what is the **converse** and what is the **inverse** of the original implication. In each case give a formula for the implication. The formula should be an implication using two of the propositions $p, q, \neg p, \neg q$.

Decide (and explain how you decided) what is:

> the contrapositive of the converse of the inverse of the contrapositive of the inverse of the converse of the inverse of the contrapositive of the inverse of the contrapositive of the converse of the inverse of the converse of the contrapositive of the inverse of the converse of the contrapositive

For the full credit you should come up with a universal rule how to decide what is the result of any statement similar to the boxed statement above.

Problem 5. This problem is inspired by Problem 42 in Section 4.3.

(a) In Problem 42 you are asked to find the number of different ways in which a horse race with 4 horses can finish if ties are possible. Let us call this number R_4. Before finding R_4, find the numbers R_1, R_2 and R_3; that is, replace 4 horses in Problem 42 by 1 horse, 2 horses and 3 horses. Now, finding R_4 will be easier. By definition set $R_0 = 1$.

(b) The goal here is to calculate R_5, R_6, R_7, R_8, and so on. Proceeding like in (a) would be a tedious task. I hope that you can discover a recursive formula that expresses R_{n+1} as a function of $R_0, R_1, \ldots, R_n$.

HINT: Let n and k be integers, $n > k$. Think how the number of different finishes of a race with k horses reflects in the number of finishes in a race with n horses. Or, with specific numbers; how the number of finishes of the races with 5 horses (R_5) relates to the number of finishes of the races with 4 horses (R_4), with 3 horses (R_3), with 2 horses (R_2), with 1 horse (R_1) and with 0 horses (R_0).

Problem 6. Let $n \in \mathbb{Z}_+$ and $n \geq 2$. This problem is about coloring a square (chess-like) board consisting of n rows and n columns for a total of n^2 squares. We will deal with only two and three colors. Each small square will be colored by one color. Similar to the chess board, just with more variety of patterns. Colors will be abbreviated as B for black, W for white and G for gray.

If $k \in \{1, \ldots, n\}$ and $\alpha \in \{B, W, G\}$ is a color, we denote by $C(k, \alpha)$ the number of squares in row k which are colored with color α. The function $C(k, \alpha)$ takes values in the set $\{0, 1, \ldots, n\}$. In fact, for a fixed coloring of an $n \times n$ board with three colors $\{B, W, G\}$ we have

$$C : \{1, \ldots, n\} \times \{B, W, G\} \to \{0, 1, \ldots, n\}.$$

In words, C is a function from the Cartesian product $\{1, \ldots, n\} \times \{B, W, G\}$ to the set $\{0, 1, \ldots, n\}$. As a practice with the function C understand that for the standard chessboard the following statement is true:

$$\forall\, k \in \{1, \ldots, 8\} \quad \forall\, \alpha \in \{B, W\} \quad C(k, \alpha) = 4.$$

(a) In this item we deal with two colors $\{B, W\}$ and n is an arbitrary integer greater than 1. Prove that there exists a coloring of an $n \times n$ board such that the following statement is true:

$$\forall\, j \in \{1, \ldots, n\} \quad \forall\, k \in \{1, \ldots, n\} \quad \forall\, \alpha \in \{B, W\} \quad j \neq k \;\Rightarrow\; C(j, \alpha) \neq C(k, \alpha).$$

(b) In this item we deal with three colors $\{B, W, G\}$ and $n = 3$. Prove that a 3×3 board can be colored with three colors in such a way that the following statement is true:

$$\forall\, j \in \{1, 2, 3\} \quad \forall\, k \in \{1, 2, 3\} \quad \forall\, \alpha \in \{B, W, G\} \quad j \neq k \;\Rightarrow\; C(j, \alpha) \neq C(k, \alpha).$$

(c) In this item we deal with three colors $\{B, W, G\}$ and $n > 3$. Prove that for an arbitrary $n \times n$ board and an arbitrary coloring of this board with three colors the following statement is true

$$\exists\, j \in \{1, \ldots, n\} \quad \exists\, k \in \{1, \ldots, n\} \quad \exists\, \alpha \in \{B, W, G\} \quad j \neq k \;\wedge\; C(j, \alpha) = C(k, \alpha).$$

(In the above logical formulas I used the symbol $\Rightarrow$ to denote the implication. Recall that the notation used in the book is $\to$. I explained why $\Rightarrow$ is preferable on the class website.)

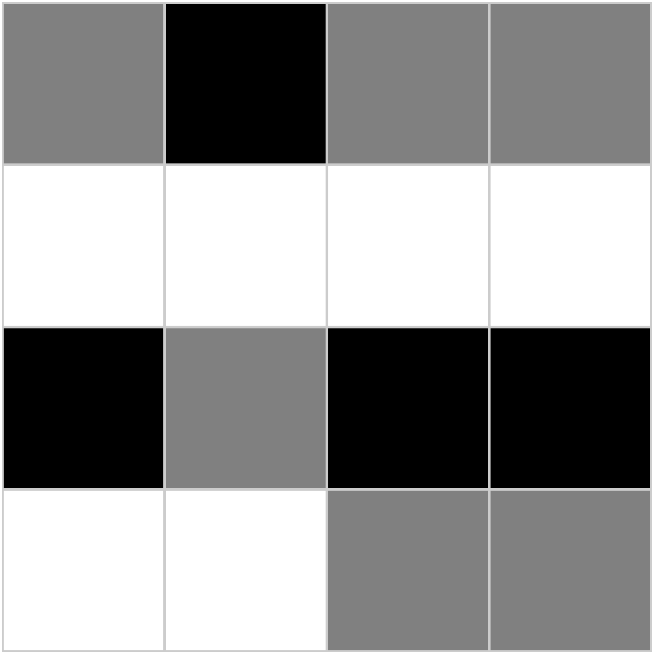

Figure 1: An example of 4×4 colored board

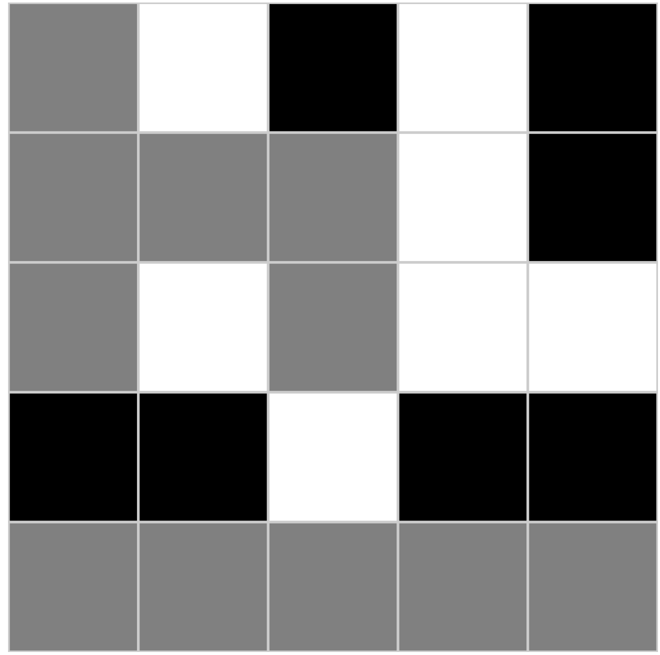

Figure 2: An example of 5×5 colored board