The Axioms for the Integers

Before stating the axioms for $\mathbb{Z}$ we make several comments about the notation.

Our textbook uses $\to$ to denote the implication, that is $p \to q$ means "p implies q". Another common notation for the implication is $\Rightarrow$. It seems more appropriate to use $\Rightarrow$ here since we use $\to$ to denote a function; like in $f : A \to B$; meaning that f is a function defined on A with the values in B.

In the axioms we use three other logical operators learned in Section 1.1: the conjunction ($\wedge$), the disjunction ($\vee$) and the exclusive disjunction ($\oplus$).

Axioms 1 and 6 below claim the existence of two functions defined on $\mathbb{Z} \times \mathbb{Z}$; the function $+ : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$, the addition, and the function $\cdot : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$, the multiplication. It is common to denote the value of $+$ at a pair $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ by $a + b$. It is also common to denote the value of $\cdot$ at a pair $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ by $a \cdot b$ which is almost always abbreviated as ab.

Axiom 1 (AE)**.** $\exists + : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$

Axiom 2 (AA)**.** $\forall a \in \mathbb{Z}\ \forall b \in \mathbb{Z}\ \forall c \in \mathbb{Z} \quad a + (b + c) = (a + b) + c$

Axiom 3 (AC)**.** $\forall a \in \mathbb{Z}\ \forall b \in \mathbb{Z} \quad a + b = b + a$

Axiom 4 (AZ)**.** $\exists 0 \in \mathbb{Z}\ \forall a \in \mathbb{Z} \quad 0 + a = a$

Axiom 5 (AO)**.** $\forall a \in \mathbb{Z}\ \exists x \in \mathbb{Z} \quad a + x = 0$

Axiom 6 (ME)**.** $\exists \cdot : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$

Axiom 7 (MA)**.** $\forall a \in \mathbb{Z}\ \forall b \in \mathbb{Z}\ \forall c \in \mathbb{Z} \quad a(bc) = (ab)c$

Axiom 8 (MC)**.** $\forall a \in \mathbb{Z}\ \forall b \in \mathbb{Z} \quad ab = ba$

Axiom 9 (MO)**.** $\exists 1 \in \mathbb{Z} \quad \big(1 \neq 0\big) \wedge \big(\forall a \in \mathbb{Z} \quad 1 \cdot a = a\big)$

Axiom 10 (MZ)**.** $\forall a \in \mathbb{Z}\ \forall b \in \mathbb{Z} \quad ab = 0 \Rightarrow (a = 0) \vee (b = 0)$

Axiom 11 (DL)**.** $\forall a \in \mathbb{Z}\ \forall b \in \mathbb{Z}\ \forall c \in \mathbb{Z} \quad a(b + c) = ab + ac$

Axiom 12 (OE)**.** $\forall a \in \mathbb{Z}\ \forall b \in \mathbb{Z} \quad (a < b) \oplus (a = b) \oplus (b < a)$

Axiom 13 (OT)**.** $\forall a \in \mathbb{Z}\ \forall b \in \mathbb{Z}\ \forall c \in \mathbb{Z} \quad (a < b) \wedge (b < c) \Rightarrow a < c$

Axiom 14 (OA)**.** $\forall a \in \mathbb{Z}\ \forall b \in \mathbb{Z}\ \forall c \in \mathbb{Z} \quad a < b \Rightarrow a + c < b + c$

Axiom 15 (OM)**.** $\forall a \in \mathbb{Z}\ \forall b \in \mathbb{Z}\ \forall c \in \mathbb{Z} \quad (a < b) \wedge (0 < c) \Rightarrow ac < bc$

Axiom 16 (WO)**.** $\big(S \subseteq \mathbb{Z}\big) \wedge \big(S \neq \emptyset\big) \wedge \big(\forall x \in S \quad 0 < x\big) \Rightarrow \big(\exists m \in S\ \forall x \in S\ (m < x) \oplus (m = x)\big)$

Explanation of abbreviations: AE - addition exists, AA - addition is associative, AC - addition is commutative, AZ - addition has zero, AO - addition has opposites, ME - multiplication exists, MA - multiplication is associative, MC - multiplication is commutative, MO - multiplication has one, MZ - multiplication respects zero, DL - distributive law, OE - order exists, OT - order is transitive, OA - order respects addition, OM - order respects multiplication, WO - the well-ordering axiom.