The Axioms for the Integers

In the axioms below we use the standard notation for logical operators: the conjunction is $\wedge$, the disjunction is $\vee$, the exclusive disjunction is $\oplus$, the implication is $\Rightarrow$, the universal quantifier is $\forall$, the existential quantifier is $\exists$.

We also use the standard set notation: the set membership $\in$, the subset $\subseteq$, the equality $=$, the set difference $\setminus$ and the Cartesian product $\times$. For singleton sets instead of writing $\{a\} = \{b\}$ we write $a = b$.

The notation $f : A \to B$ stands for a function f which is defined on a set A with the values in B.

Axiom 2 below establishes the existence of the addition function defined on $\mathbb{Z}\times\mathbb{Z}$ with the values in $\mathbb{Z}$. It is common to denote the value of $+$ at a pair $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ by $a + b$.

Axiom 7 establishes the existence of the multiplication function defined on $\mathbb{Z}\times\mathbb{Z}$ with the values in $\mathbb{Z}$. It is common to denote the value of this function at a pair $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ by $a \cdot b$ which is almost always abbreviated as ab.

Axiom 12 introduces the set of positive integers.

Definition. The set $\mathbb{Z}$ of *integers* satisfies the following 16 axioms.

Axiom 1 (ZE). $\mathbb{Z} \neq \emptyset$

Axiom 2 (AE). $\exists \ + : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$

Axiom 3 (AA). $\forall a \in \mathbb{Z} \ \ \forall b \in \mathbb{Z} \ \ \forall c \in \mathbb{Z} \quad a + (b + c) = (a + b) + c$

Axiom 4 (AC). $\forall a \in \mathbb{Z} \ \ \forall b \in \mathbb{Z} \quad a + b = b + a$

Axiom 5 (AZ). $\exists 0 \in \mathbb{Z} \ \ \forall a \in \mathbb{Z} \quad 0 + a = a$

Axiom 6 (AO). $\forall a \in \mathbb{Z} \ \ \exists (-a) \in \mathbb{Z} \quad a + (-a) = 0$

Axiom 7 (ME). $\exists \ \cdot : \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}$.

Axiom 8 (MA). $\forall a \in \mathbb{Z} \ \ \forall b \in \mathbb{Z} \ \forall c \in \mathbb{Z} \quad a(bc) = (ab)c$

Axiom 9 (MC). $\forall a \in \mathbb{Z} \ \ \forall b \in \mathbb{Z} \quad ab = ba$

Axiom 10 (MO). $\exists 1 \in \mathbb{Z}\setminus\{0\} \quad \forall a \in \mathbb{Z} \quad 1 \cdot a = a$

Axiom 11 (DL). $\forall a \in \mathbb{Z} \ \ \forall b \in \mathbb{Z} \ \ \forall c \in \mathbb{Z} \quad a(b + c) = ab + ac$

Axiom 12 (PE). $\exists \ \mathbb{P} \quad \big(\mathbb{P} \subseteq \mathbb{Z}\setminus\{0\}\big) \wedge \big(\mathbb{P} \neq \emptyset\big)$

Axiom 13 (PD). $\forall a \in \mathbb{Z}\setminus\{0\} \quad (a \in \mathbb{P}) \oplus (-a \in \mathbb{P})$

Axiom 14 (PA). $\forall a \in \mathbb{P} \ \ \forall b \in \mathbb{P} \quad a + b \in \mathbb{P}$

Axiom 15 (PM). $\forall\, a \in \mathbb{P} \ \ \forall b \in \mathbb{P} \quad ab \in \mathbb{P}$

Axiom 16 (WO). $\big(S \subseteq \mathbb{P}\big) \wedge \big(S \neq \emptyset\big) \Rightarrow \big(\exists\, m \in S \ \ \forall x \in S\setminus\{m\} \quad x + (-m) \in \mathbb{P}\big)$

Explanation of abbreviations: ZE - integers exist, AE - addition exists, AA - addition is associative, AC - addition is commutative, AZ - addition has zero, AO - addition has opposites, ME - multiplication exists, MA - multiplication is associative, MC - multiplication is commutative, MO - multiplication has one, MZ - multiplication respects zero, DL - distributive law, PE - positive integers exist, PD - dichotomy involving positive integers, PA - positive integers respect addition, PM - positive integers respect multiplication, WO - the well-ordering axiom.