MATH 209

Examination 2h
August 6, 2012

Name ______________________

1. (a) Prove that if x is a real number, then $\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor$.

   (b) Prove that if x is a real number, then $\lceil 2x \rceil = \lceil x \rceil + ???$. (You invent the formula and prove it.)

   (c) Prove that if x is a real number, then $\lceil 3x \rceil = \lceil x \rceil + \lceil x - \frac{1}{3} \rceil + \lceil x - \frac{2}{3} \rceil$.

2. (a) State the definition of a countable set. (You must use the word "bijection" in this definition.)

   (b) Prove that the set $\mathbb{Z}$ of all integers is countable. (You need to prove that the formula that you are giving is really a bijection.)

3. Recall that the factorial of a nonnegative integer is recursively defined by

$$0! = 1, \qquad \forall n \in \mathbb{Z}_+ \;\; n! = n \cdot (n-1)!.$$

   (a) Prove

$$\forall n \in \mathbb{Z}_+ \qquad \frac{1}{(n+1)!} \leq \frac{1}{n(n+1)}.$$

   (b) Prove

$$\forall n \in \mathbb{Z}_+ \qquad \sum_{j=0}^{n} \frac{1}{j!} \leq 3 - \frac{1}{n}.$$

4. Recall that the Fibonacci numbers are recursively defined by

$$f_0 = 0, \quad f_1 = 1, \qquad \forall n \in \mathbb{Z}_+ \;\; f_{n+1} = f_n + f_{n-1}.$$

   Prove

$$\forall n \in \mathbb{Z}_+ \qquad \sum_{j=1}^{n} j f_j = n f_{n+2} - f_{n+3} + 2.$$