MATH 209

Final Examination
March 15, 2011

Name ______________________

1. (a) Is it possible for the propositions $p \vee q$ and $\neg p \vee \neg q$ to be both false? Justify your answer.
   (b) Is it possible for the proposition $p \to \bigl(\neg p \to q\bigr)$ to be false? Justify your answer.
   (c) Prove or disprove: $\bigl((p \to q) \to r\bigr) \to \bigl((r \to q) \to p\bigr)$ is a tautology.

2. The universe of discourse in this problem is the set of all integers. Consider the following three statements.

   (a) $\forall x\ \forall y\ \bigl(x^2 = y^2 \to x = y\bigr)$, (b) $\exists x\ \forall y\ \bigl(x^2 = y^2 \to x = y\bigr)$, (c) $\exists x\ \forall y\ \bigl(xy \geq x\bigr)$.

   Write the negation of each of these statements. Decide and state clearly which statements are true. Prove the statements which are true.

3. Let x and y be real numbers. Determine all possible values for $\lceil x + y \rceil$ in terms of $\lceil x \rceil$ and $\lceil y \rceil$. Illustrate all possible cases with some famous numbers (e.g., $\pi, e, \sqrt{2}, \sqrt{3}, \ldots$) as examples. Justify that all possible cases are included in your list.

4. (a) Let $S = \{a, b, c, d\}$. Define a specific bijection between the power set $P(S)$ and the set of all bit strings of length 4. (This bijection should be "logical" so that you can use it to answer (4c) below. Hint: $f(\emptyset) = 0000, f(S) = 1111$.)
   (b) What is the cardinality of the power set $P(S)$?
   (c) If a set has n elements, what is the cardinality of its power set? Prove your claim.

5. Let r be a real number such that $r \neq 0$ and $r \neq 1$. Let n be a nonnegative integer. State and prove the closed form expression formula for the geometric sum

   $$\sum_{j=0}^{n} r^j = 1 + r + \cdots + r^n.$$

   Hint: If you cannot remember this formula you might be able to guess it for $r = 2$ and $r = 1/2$. Then try to guess the general formula. If you do not succeed, then prove the formula for $r = 2$.

6. Define a sequence a_n recursively by: $a_0 = 1, \qquad a_{n+1} = \displaystyle\sum_{j=0}^{n} a_j = a_0 + \cdots + a_n, \quad n \in \mathbb{N}.$
   (a) Compute $a_0, a_1, a_2, a_3, a_4, a_5$.
   (b) Use strong induction to prove that $a_n = 2^{n-1}$ for all positive integers n.

7. (a) How many bit strings of length 9 <u>do not contain</u> the pattern 00. (b) Based on the calculation in (a) count how many bit strings of length 9 contain at least one occurrence of the pattern 00 and at least one occurrence of 11. (Hints: (a) Place 1s first; how many; you decide. (b) Look at the complement; it is an inclusion-exclusion problem.)

8. (a) How many different strings can be made from the letters in REARRANGE, using all the letters?
   (b) How many ways are there to rearrange the letters in REARRANGE into two separate words? (such as: GREEN REAR)

9. This is a "wallet" problem. Consider the equation $x_1 + x_2 + x_3 = 24$.

   (a) How many triples (x_1, x_2, x_3) of nonnegative integers satisfy the given equation ?
   (b) How many triples (x_1, x_2, x_3) of positive integers satisfy the given equation?
   (c) How many triples (x_1, x_2, x_3) of digits, that is $x_1, x_2, x_3 \in \bigl\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\bigr\}$, satisfy the given equation? (There are not too many triples here; you can even count them all.)

10. Each student in a class of 28 chooses 14 other students in the class and sends each one an email. Prove that some pair of students must send each other emails.