# MATH 209

Final Examination
June 6, 2012

Name ____________________

1. We are very well familiar with the sequence $f_n$ of the Fibonacci numbers. Here I list them, but I extend the sequence towards left following the same pattern.

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | $f_0$ | $f_1$ | $f_2$ | $f_3$ | $f_4$ | $f_5$ | $\cdots$ |
| $\cdots$ | 5 | $-3$ | 2 | $-1$ | 1 | 0 | 1 | 1 | 2 | 3 | 5 | $\cdots$ |
| $\cdots$ | $a_5$ | $a_4$ | $a_3$ | $a_2$ | $a_1$ | $a_0$ | | | | | | |

   (a) Find the recursive relation for the sequence $a_0, a_1, \ldots, a_n, \ldots$.

   (b) Find the formula for $a_n$ in terms of $f_n$ (this formula should hold for all $n \in \mathbb{N}$). Prove your formula using mathematical induction.

2. Let $n \in \mathbb{N}$. Denote by $s_n$ the number of bit strings of length $n$ that contain at least one occurrence of 01.

   (a) Calculate $s_0, s_1, s_2, s_3, s_4$.

   (b) Find a recurrence relation for the sequence $s_n$.

   (c) Use the complement rule to give a formula for $s_n$ in terms of $n$.

3. In this problem we consider the equation $x_1 + x_2 + x_3 + x_4 = 10$.

   (a) How many nonnegative integer solutions does this equation have?

   (b) How many of those solutions include digits only, that is such that $0 \leq x_j \leq 9, j = 1, 2, 3, 4$?

   (c) How many non negative solutions consists of even numbers only?

4. Let $n \in \mathbb{Z}_+$ and $n \geq 3$. Prove the identity $\binom{2n}{3} = 2n\binom{n}{2} + 2\binom{n}{3}$ in two different ways: using a combinatorial argument and by algebraic manipulation.

5. The year 2012 is a **leap year**. A year $Y$ is a **leap year** if $Y$ is divisible by 4 but not divisible by 100 unless $Y$ is divisible by 400. State three logical statements mentioned in the definition of a leap year and name them $p, q, r$. Then write the definition of the leap year as a compound logical statement using $p, q$ and $r$ and $\wedge, \vee, \neg$ or any other logical operations.

6. Let $S = \{s_1, s_2, s_3, s_4, s_5\}$ be a set of five distinct shirts and let $P = \{p_1, p_2, p_3, p_4\}$ be a set of four distinct pants. The table below contains outfits worn during a week.

   (a) Do the listed outfits define a function from $S$ to $P$? Why?

   (b) Do the listed outfits define a function from $P$ to $S$? Why?

Outfits:

| day | M | T | W | R | F | Sa | Su |
|---|---|---|---|---|---|---|---|
| shirt | $s_2$ | $s_3$ | $s_1$ | $s_4$ | $s_3$ | $s_5$ | $s_4$ |
| pants | $p_3$ | $p_2$ | $p_1$ | $p_4$ | $p_1$ | $p_2$ | $p_2$ |

   Is it possible to select days of the week so that the outfits worn during those days do define:

   (c) a function from $S$ to $P$? Explain?

   (d) an injection from $S$ to $P$? Explain?

   (e) a surjection from $S$ to $P$? Explain?

   (f) a function from $P$ to $S$? Explain?

   (g) an injection from $P$ to $S$? Explain?

   (h) a surjection from $P$ to $S$? Explain?

7. Consider a recursively defined sequence $p_0 = 1, \quad p_n = \sqrt{1 + p_{n-1}}, \quad n \in \mathbb{Z}_+$. Prove the following statement: $\forall\, n \in \mathbb{Z}_+ \; p_n$ is irrational.

8. There are two counting questions in this problem. They should both lead to the same numerical answer. The setting of this problem is illustrated with two pictures on the back.

   (a) How many different ways are there to color a $3 \times 3$ square board with three colors, say Black, Gray and White, if you are allowed to use each color on exactly three squares?

   (b) How many ways are there to travel in $xyz$ space from the origin $(0, 0, 0)$ to the point $(3, 3, 3)$ by taking steps one unit in the positive $x$ direction, one unit in the positive $y$ direction, or one unit in the positive $z$ direction? (Moving in the negative $x$, $y$, or $z$ direction is prohibited, so that no backtracking is allowed.)

   (c) Explain why the answers in (8a) and (8b) are the same. This should be done by establishing a bijection from the set described in (8a) to the set described in (8b).

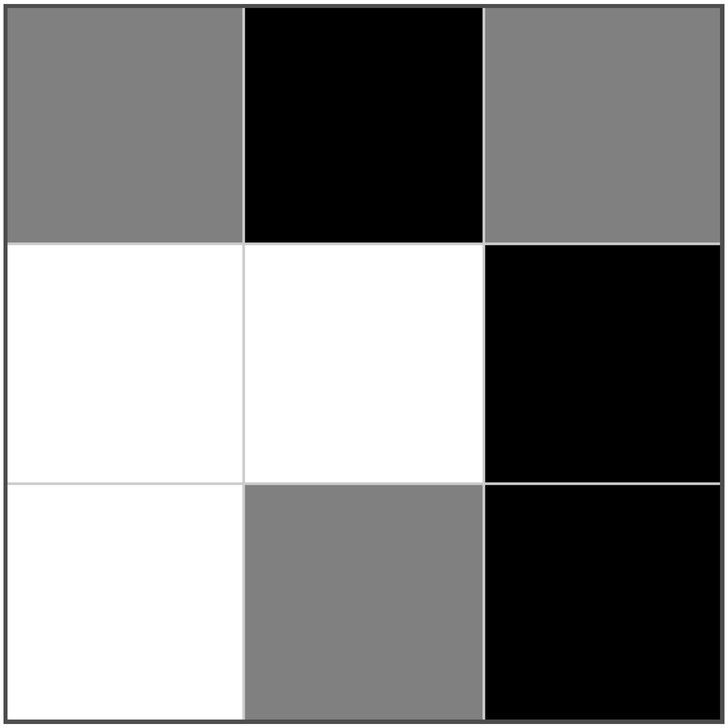

Figure 1: A colored $3 \times 3$ board

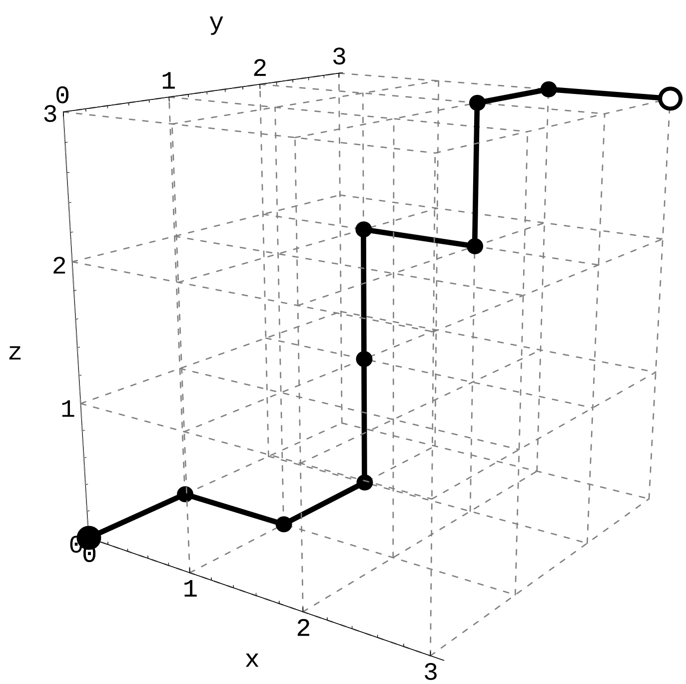


Figure 2: A path in a $3 \times 3 \times 3$ cube