Section 15.2 Problem 14. (a) Compute the critical points of $f(x,y) = 2x^2 - 3xy + 8y^2 + x - y$ and classify them.
(b) By completing the square, plot the contour diagram of f and show that the local extremum found in part (a) is a global one.

Solution. First find the partial derivatives $f_x(x,y) = 4x-3y+1$ and $f_y(x,y) = -3x+16y-1$. Then solve

$$\begin{aligned} 4x - 3y &= -1 \\ -3x + 16y &= 1 \end{aligned}$$

to get the critical point $x = -13/55, y = 1/55$. Now find the value of f at the critical point to be $-7/55$.

How to complete the square? The idea is from the book on page 765. For the function $au^2 + buv + cv^2$ the squares can be completed as

$$au^2 + buv + cv^2 = a\left(u + \frac{b}{2a}v\right)^2 + \frac{4ac - b^2}{4a}v^2$$

Specifically

$$2u^2 - 3uv + 8v^2 = 2\left(u - \frac{3}{4}v\right)^2 + \frac{55}{8}v^2$$

But, how is the formula for $f(x,y)$ related to the formula involving u and v? In the formula involving u and v the critical point is at the origin $u = 0, v = 0$. So, the idea here is to move the critical point of $f(x,y)$ to the origin. For that purpose we introduce the new variables u and v by

$$u = x + \frac{13}{55}, \qquad v = y - \frac{1}{55}.$$

Now substitute

$$x = u - \frac{13}{55}, \qquad y = v + \frac{1}{55}$$

in the formula for f to get a new function of u and v

$$\begin{aligned} f(x,y) &= f\left(u - \tfrac{13}{55}, v + \tfrac{1}{55}\right) \\ &= 2\left(u - \tfrac{13}{55}\right)^2 - 3\left(u - \tfrac{13}{55}\right)\left(v + \tfrac{1}{55}\right) + 8\left(v + \tfrac{1}{55}\right)^2 + \left(u - \tfrac{13}{55}\right) - \left(v + \tfrac{1}{55}\right) \\ &= 2u^2 - \tfrac{52}{55}u + 2\left(\tfrac{13}{55}\right)^2 - 3uv + \tfrac{39}{55}v - \tfrac{3}{55}u + \tfrac{39}{55^2} + 8v^2 + \tfrac{16}{55}v + 8\left(\tfrac{1}{55}\right)^2 + u - \tfrac{13}{55} - v - \tfrac{1}{55} \\ &= 2u^2 - 3uv + 8v^2 + \tfrac{39+2*169+8-13*55-55}{55^2} \\ &= 2u^2 - 3uv + 8v^2 - \tfrac{7}{55} \end{aligned}$$

Now we use the completed square formula for $2u^2 - 3uv + 8v^2$ derived above to continue:

$$= 2\left(u - \frac{3}{4}v\right)^2 + \frac{55}{8}v^2 - \frac{7}{55}$$

Now we go back to the variables x, y:

$$
\begin{aligned}
&= 2\left(\left(x+\tfrac{13}{55}\right)-\frac{3}{4}\left(y-\tfrac{1}{55}\right)\right)^2+\frac{55}{8}\left(y-\tfrac{1}{55}\right)^2-\frac{7}{55}\\
&= 2\left(x+\tfrac{13}{55}-\tfrac{3}{4}\,y+\tfrac{3}{4}\tfrac{1}{55}\right)^2+\frac{55}{8}\left(y-\tfrac{1}{55}\right)^2-\frac{7}{55}\\
&= 2\left(x-\tfrac{3}{4}\,y+\tfrac{52+3}{4*55}\right)^2+\frac{55}{8}\left(y-\tfrac{1}{55}\right)^2-\frac{7}{55}\\
&= 2\left(x-\frac{3}{4}\,y+\frac{1}{4}\right)^2+\frac{55}{8}\left(y-\frac{1}{55}\right)^2-\frac{7}{55}
\end{aligned}
$$

Since in the last formula two squares are added to $-7/55$ it is clear that the smallest value of $f(x,y)$ is $-7/55$. It is also important to notice that for $x = -13/55$ and $y = 1/55$ two squares evaluate to 0.

And there is a different, more direct way to get to the same formula. First complete the square of the part which involves only x

$$
\begin{aligned}
2x^2+x(1-3y) &= 2\left(x^2+\frac{1}{2}x(1-3y)\right)\\
&= 2\left(x^2+2\frac{1}{4}x(1-3y)+\frac{1}{16}(1-3y)^2\right)-2\frac{1}{16}(1-3y)^2\\
&= 2\left(x+\frac{1}{4}(1-3y)\right)^2-\frac{1}{8}(1-3y)^2
\end{aligned}
$$

Now we use this in the formula for $f(x,y)$:

$$
\begin{aligned}
2x^2-3xy+8y^2+x-y &= 2\left(x+\frac{1}{4}(1-3y)\right)^2-\frac{1}{8}(1-3y)^2+8y^2-y\\
&= 2\left(x+\frac{1}{4}(1-3y)\right)^2+\frac{55}{8}y^2-\frac{1}{4}y-\frac{1}{8}\\
&= 2\left(x+\frac{1}{4}(1-3y)\right)^2+\frac{55}{8}\left(y^2-\frac{2}{55}y-\frac{1}{55}\right)\\
&= 2\left(x+\frac{1}{4}(1-3y)\right)^2+\frac{55}{8}\left(\left(y-\frac{1}{55}\right)^2-\frac{56}{55^2}\right)\\
&= 2\left(x+\frac{1}{4}(1-3y)\right)^2+\frac{55}{8}\left(y-\frac{1}{55}\right)^2-\frac{7}{55}
\end{aligned}
$$