Section 15.2 Problem 40. Find the minimum distance from the point $(1, 2, 10)$ to the paraboloid given by the equation $z = x^2 + y^2$.

Solution. We need to minimize the function

$$f(x, y, z) = (x - 1)^2 + (y - 2)^2 + (z - 10)^2$$

subject to the constraint

$$g(x, y, z) = x^2 + y^2 - z.$$

We set up the equations

$$\begin{aligned} x^2 + y^2 - z &= 0 \\ 2(x - 1) &= 2\lambda x \\ 2(y - 2) &= 2\lambda y \\ 2(z - 10) &= -\lambda \end{aligned}$$

We use the last three equations to express x, y, z in terms of λ

$$x = \frac{1}{1 - \lambda}, \quad y = \frac{2}{1 - \lambda}, \quad z = 10 - \frac{\lambda}{2}$$

Substituting in the first equation and simplifying we get

$$\lambda^3 - 22\lambda^2 + 41\lambda - 10 = 0.$$

The exact expressions for the roots of this equation are too complicated. The approximations for the roots are

$$\lambda_1 \approx 0.28775, \quad \lambda_2 \approx 1.74, \quad \lambda_3 \approx 19.972$$

The corresponding points on the paraboloid are:

$$(1.404, 2.808, 9.8561), \quad (-1.3513, -2.7026, 9.13), \quad (-0.052709, -0.10542, 0.013891)$$

Now, only one of these three points is closest to the given point $(1, 2, 10)$. A simple inspection yields that the first point $(1.404, 2.808, 9.8561)$ is the closest one. To verify that numerically we would calculate the value of the function $f(x, y, z)$ at each of these points:

$$\begin{aligned} f(1.404, 2.808, 9.8561) &\approx 0.83679 \\ f(-1.3513, -2.7026, 9.13) &\approx 28.4 \\ f(-0.052709, -0.10542, 0.013891) &\approx 105.26 \end{aligned}$$

This confirms that indeed the closest point to $(1, 2, 10)$ on the paraboloid is the point $(1.404, 2.808, 9.8561)$.

You can try replacing the point $(1, 2, 10)$ with the point $(-1, -2, 6)$, or the point $(-3, -6, 7)$, or the point $(-5, -10, 8)$. Then, the resulting cubic equation can be solved symbolically.

Remark. Approximations are presented rounded to 5 significant digits.