Chapter 15 Review Problem 46. An irrigation canal has trapezoidal cross section of area A. Minimize the perimeter p.

Solution. In the book they give three variables: d, w and θ. I think that it is easier to replace θ with another variable c where

$$c = \frac{d}{\tan\theta}$$

To visualize c in the picture draw a right triangle around θ. One of its sides is d the other is c.
Now the area A is given by

$$(w + c)d = A$$

The perimeter p to be minimized is

$$p = 2\sqrt{d^2 + c^2} + w.$$

Applying the method of Lagrange multipliers we get the equations

$$(w + c)d = A \tag{1}$$

$$\frac{2d}{\sqrt{d^2 + c^2}} = \lambda(w + c) \tag{2}$$

$$\frac{2c}{\sqrt{d^2 + c^2}} = \lambda d \tag{3}$$

$$1 = \lambda d \tag{4}$$

Dividing equation (2) by (3) and substituting $w + c = A/d$ (from (1)) we get

$$\frac{d}{c} = \frac{w + c}{d} = \frac{A}{d^2} \tag{5}$$

Substituting equation (4) in (3) and simplifying we get

$$4c^2 = d^2 + c^2, \qquad \text{or} \qquad 3c^2 = d^2.$$

Therefore

$$\frac{d}{c} = \sqrt{3}. \tag{6}$$

Substituting this in (5) we get

$$\sqrt{3} = \frac{A}{d^2}.$$

Hence

$$d = \sqrt{\frac{A}{\sqrt{3}}} = \frac{1}{\sqrt[4]{3}}\sqrt{A}.$$

From (6) we get

$$c = \frac{1}{\sqrt{3}}d = \frac{1}{\sqrt{3\sqrt{3}}}\sqrt{A}$$

From (5) we get

$$w = \frac{d}{c}d - c = \sqrt{3}d - \frac{1}{\sqrt{3}}d = \frac{2}{\sqrt{3}}d = 2c = \frac{2}{\sqrt{3\sqrt{3}}}\sqrt{A}.$$

Now calculate θ:

$$\theta = \arctan\frac{d}{c} = \arctan\sqrt{3} = \frac{\pi}{3}.$$

Now calculate the length of the slanted side s of the canal:

$$s = \sqrt{d^2 + c^2} = \sqrt{3c^2 + c^2} = \sqrt{4c^2} = 2c = w.$$

Thus the most efficient canal is one with the slanted side equal to the base.

The last discovery inspires us to approach the problem somewhat differently. Introduce the new variable s, the slanted side of the canal:

$$s = \frac{d}{\sin\theta} = \sqrt{d^2 + c^2}.$$

We will solve the problem now with the variables s, w and θ. The formula for the area is somewhat complicated:

$$A = s\,\sin\theta\big(s\cos\theta + w\big)$$

The function to be minimized is simpler:

$$2s + w$$

Applying the method of Lagrange multipliers we get the equations

$$s\,\sin\theta\big(s\cos\theta + w\big) = A \tag{7}$$

$$1 = \lambda\, s\,\sin\theta \tag{8}$$

$$2 = \lambda\left(\sin\theta\big(s\cos\theta + w\big) + s\sin\theta\cos\theta\right) \tag{9}$$

$$0 = \lambda\big(s\,\cos\theta\big(s\cos\theta + w\big) - s^2(\sin\theta)^2\big) \tag{10}$$

Dividing equation (9) by (8) we get

$$2 = 2\cos\theta + \frac{w}{s} \tag{11}$$

Dividing equation (10) by λs^2 we get

$$0 = (\cos\theta)^2 + \frac{w}{s}\cos\theta - (\sin\theta)^2 = 2(\cos\theta)^2 - 1 + \frac{w}{s}\cos\theta. \tag{12}$$

Substituting w/s from (11) into (12) we get

$$0 = 2(\cos\theta)^2 - 1 + (2 - 2\cos\theta)\cos\theta = -1 + 2\cos\theta$$

The last equation is easily solved

$$\cos\theta = \frac{1}{2}, \qquad \text{that is} \qquad \theta = \frac{\pi}{3}.$$

Consequently

$$\frac{w}{s} = 2 - 2\cos\theta = 1.$$

Thus $w = s$. We calculate s from (7):

$$A = s\frac{\sqrt{3}}{2}\big(s/2 + s\big) = s^2\frac{3\sqrt{3}}{4}.$$

As before

$$s = w = \frac{2}{\sqrt{3\sqrt{3}}}\sqrt{A}.$$
