

### Formula for a Production Function

Production functions are often approximated by formulas of the form

$$P = f(N, V) = cN^\alpha V^\beta$$

where  $P$  is the quantity produced and  $c$ ,  $\alpha$ , and  $\beta$  are positive constants,  $0 < \alpha < 1$  and  $0 < \beta < 1$ .

**Example 8** Show that the contours of the function  $P = cN^\alpha V^\beta$  have approximately the shape of the contours in Figure 12.48.

**Solution** The contours are the curves where  $P$  is equal to a constant value, say  $P_0$ , that is, where

$$cN^\alpha V^\beta = P_0.$$

Solving for  $V$  we get

$$V = \left(\frac{P_0}{c}\right)^{1/\beta} N^{-\alpha/\beta}.$$

Thus,  $V$  is a power function of  $N$  with a negative exponent, so its graph has the shape shown in Figure 12.48.

### The Cobb-Douglas Production Model

In 1928, Cobb and Douglas used a similar function to model the production of the entire US economy in the first quarter of this century. Using government estimates of  $P$ , the total yearly production between 1899 and 1922, of  $K$ , the total capital investment over the same period, and of  $L$ , the total labor force, they found that  $P$  was well approximated by the *Cobb-Douglas production function*

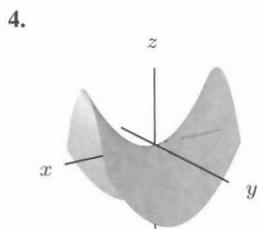
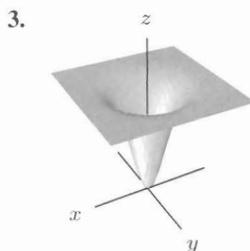
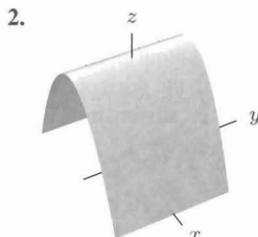
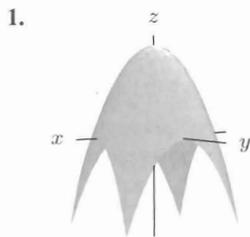
$$P = 1.01L^{0.75}K^{0.25}.$$

This function turned out to model the US economy surprisingly well, both for the period on which it was based, and for some time afterward.

## Exercises and Problems for Section 12.3

### Exercises

For each of the surfaces in Exercises 1–4, sketch a possible contour diagram, marked with reasonable  $z$ -values. (Note: There are many possible answers.)



In Exercises 5–13, sketch a contour diagram for the function with at least four labeled contours. Describe in words the contours and how they are spaced.

5.  $f(x, y) = x + y$

6.  $f(x, y) = 3x + 3y$

7.  $f(x, y) = x^2 + y^2$

8.  $f(x, y) = -x^2 - y^2 + 1$

9.  $f(x, y) = xy$

10.  $f(x, y) = y - x^2$

11.  $f(x, y) = x^2 + 2y^2$

12.  $f(x, y) = \sqrt{x^2 + 2y^2}$

13.  $f(x, y) = \cos \sqrt{x^2 + y^2}$

14. Find an equation for the contour of  $f(x, y) = 3x^2y + 7x + 20$  that goes through the point  $(5, 10)$ .

15. (a) For  $z = f(x, y) = xy$ , sketch and label the level curves  $z = \pm 1$ ,  $z = \pm 2$ .

(b) Sketch and label cross-sections of  $f$  with  $x = \pm 1$ ,  $x = \pm 2$ .

(c) The surface  $z = xy$  is cut by a vertical plane containing the line  $y = x$ . Sketch the cross-section.

16. Match the surfaces (a)–(e) in Figure 12.49 with the contour diagrams (I)–(V) in Figure 12.50.

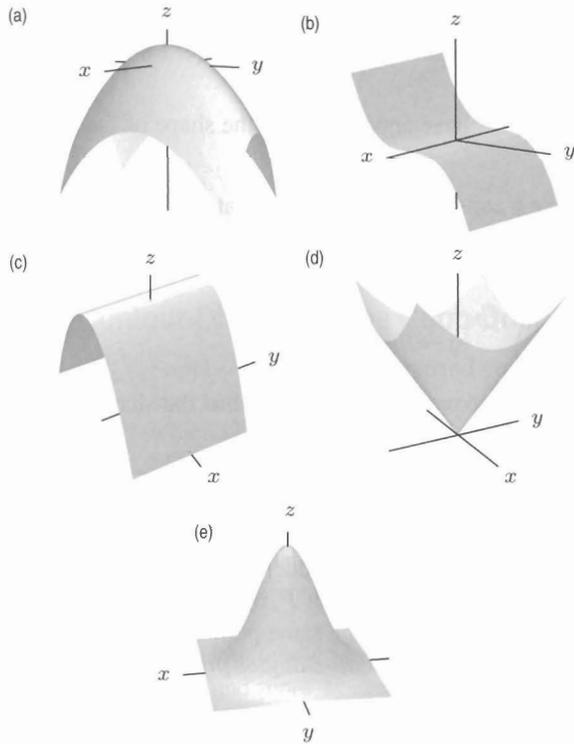


Figure 12.49

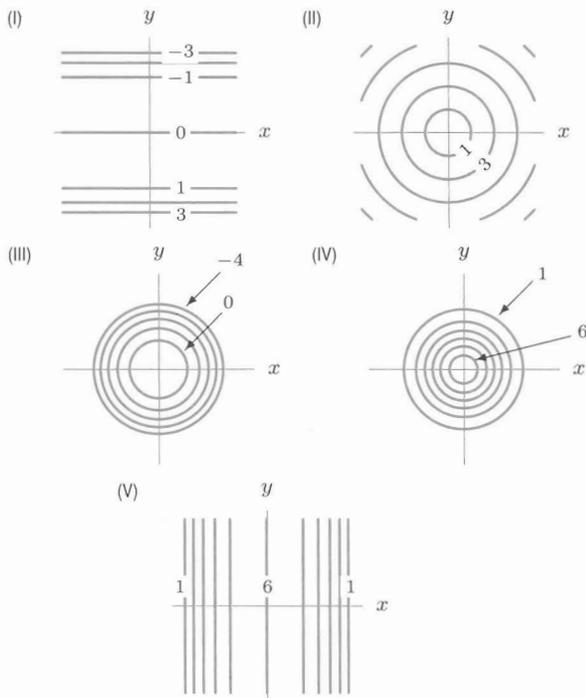


Figure 12.50

17. Match Tables 12.5–12.8 with the contour diagrams (I)–(IV) in Figure 12.51.

Table 12.5

$y \backslash x$	-1	0	1
-1	2	1	2
0	1	0	1
1	2	1	2

Table 12.6

$y \backslash x$	-1	0	1
-1	0	1	0
0	1	2	1
1	0	1	0

Table 12.7

$y \backslash x$	-1	0	1
-1	2	0	2
0	2	0	2
1	2	0	2

Table 12.8

$y \backslash x$	-1	0	1
-1	2	2	2
0	0	0	0
1	2	2	2

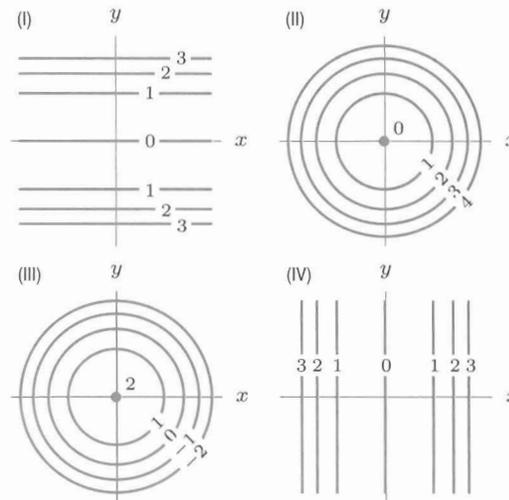


Figure 12.51

18. Total sales,  $Q$ , of a product is a function of its price and the amount spent on advertising. Figure 12.52 shows a contour diagram for total sales. Which axis corresponds to the price of the product and which to the amount spent on advertising? Explain.

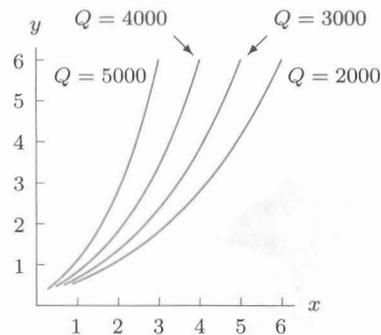


Figure 12.52

## Problems

19. Figure 12.53 shows contours of  $f(x, y) = 100e^x - 50y^2$ . Find the values of  $f$  on the contours. They are equally spaced multiples of 10.

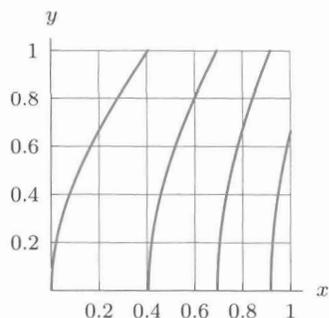


Figure 12.53

20. Figure 12.54 shows the level curves of the temperature  $H$  in a room near a recently opened window. Label the three level curves with reasonable values of  $H$  if the house is in the following locations.

- Minnesota in winter (where winters are harsh).
- San Francisco in winter (where winters are mild).
- Houston in summer (where summers are hot).
- Oregon in summer (where summers are mild).



Figure 12.54

21. Figure 12.55 shows a contour map of a hill with two paths,  $A$  and  $B$ .

- On which path,  $A$  or  $B$ , will you have to climb more steeply?
- On which path,  $A$  or  $B$ , will you probably have a better view of the surrounding countryside? (Assuming trees do not block your view.)
- Alongside which path is there more likely to be a stream?

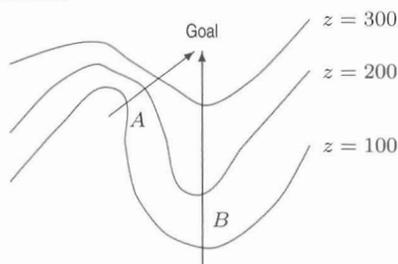


Figure 12.55

22. Figure 12.56 is a contour diagram of the monthly payment on a 5-year car loan as a function of the interest rate and the amount you borrow. The interest rate is 13% and you borrow \$6000.

- What is your monthly payment?
- If interest rates drop to 11%, how much more can you borrow without increasing your monthly payment?
- Make a table of how much you can borrow, without increasing your monthly payment, as a function of the interest rate.

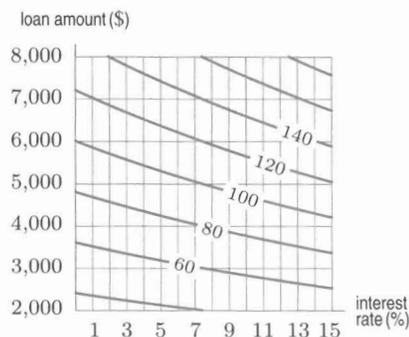


Figure 12.56

23. Describe in words the level surfaces of the function  $g(x, y, z) = \cos(x + y + z)$ .
24. Match the functions (a)–(d) with the shapes of their level curves (I)–(IV). Sketch each contour diagram.

- $f(x, y) = x^2$
- $f(x, y) = x^2 + 2y^2$
- $f(x, y) = y - x^2$
- $f(x, y) = x^2 - y^2$

- Lines
- Parabolas
- Hyperbolas
- Ellipses

25. Figure 12.57 shows the density of the fox population  $P$  (in foxes per square kilometer) for southern England. Draw two different cross-sections along a north-south line and two different cross-sections along an east-west line of the population density  $P$ .

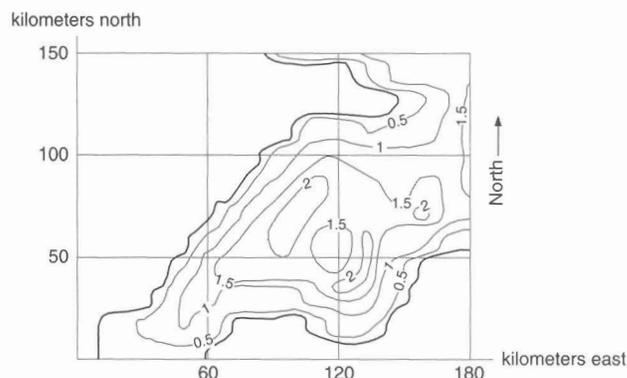


Figure 12.57

26. A manufacturer sells two goods, one at a price of \$3000 a unit and the other at a price of \$12,000 a unit. A quantity  $q_1$  of the first good and  $q_2$  of the second good are sold at a total cost of \$4000 to the manufacturer.

- (a) Express the manufacturer's profit,  $\pi$ , as a function of  $q_1$  and  $q_2$ .
- (b) Sketch curves of constant profit in the  $q_1 q_2$ -plane for  $\pi = 10,000$ ,  $\pi = 20,000$ , and  $\pi = 30,000$  and the break-even curve  $\pi = 0$ .

27. Match each Cobb-Douglas production function (a)–(c) with a graph in Figure 12.58 and a statement (D)–(G).

- (a)  $F(L, K) = L^{0.25} K^{0.25}$
- (b)  $F(L, K) = L^{0.5} K^{0.5}$
- (c)  $F(L, K) = L^{0.75} K^{0.75}$
- (D) Tripling each input triples output.
- (E) Quadrupling each input doubles output.
- (G) Doubling each input almost triples output.

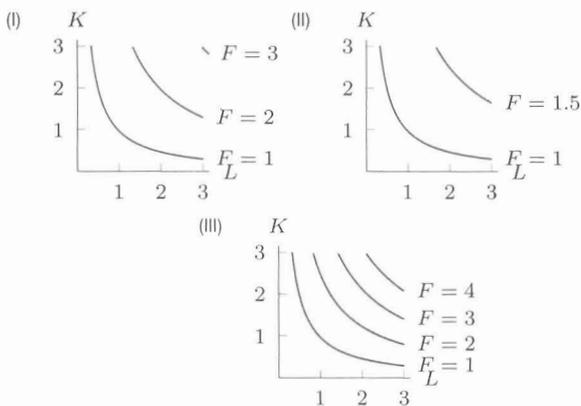


Figure 12.58

28. A general Cobb-Douglas production function has the form

$$P = cL^\alpha K^\beta.$$

What happens to production if labor and capital are both scaled up? For example, does production double if both labor and capital are doubled? Economists talk about

- *increasing returns to scale* if doubling  $L$  and  $K$  more than doubles  $P$ ,
- *constant returns to scale* if doubling  $L$  and  $K$  exactly doubles  $P$ ,
- *decreasing returns to scale* if doubling  $L$  and  $K$  less than doubles  $P$ .

What conditions on  $\alpha$  and  $\beta$  lead to increasing, constant, or decreasing returns to scale?

29. Figure 12.59 is the contour diagram of  $f(x, y)$ . Sketch the contour diagram of each of the following functions.

- (a)  $3f(x, y)$
- (b)  $f(x, y) - 10$
- (c)  $f(x - 2, y - 2)$
- (d)  $f(-x, y)$

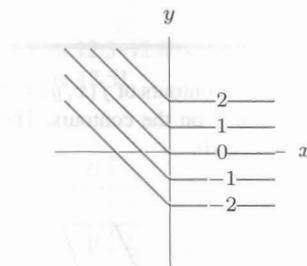


Figure 12.59

30. Figure 12.60 shows part of the contour diagram of  $f(x, y)$ . Complete the diagram for  $x < 0$  if

- (a)  $f(-x, y) = f(x, y)$
- (b)  $f(-x, y) = -f(x, y)$

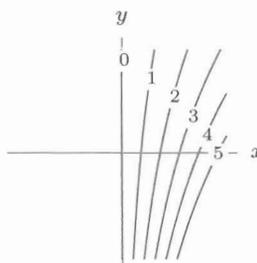


Figure 12.60

31. Values of  $f(x, y) = \frac{1}{2}(x + y - 2)(x + y - 1) + y$  are in Table 12.9.

- (a) Find a pattern in the table. Make a conjecture and use it to complete Table 12.9 without computation. Check by using the formula for  $f$ .
- (b) Using the formula, check that the pattern holds for all  $x \geq 1$  and  $y \geq 1$ .

Table 12.9

		$y$					
		1	2	3	4	5	6
$x$	1	1	3	6	10	15	21
	2	2	5	9	14	20	
	3	4	8	13	19		
	4	7	12	18			
	5	11	17				
	6	16					

32. The temperature  $T$  (in  $^{\circ}\text{C}$ ) at any point in the region  $-10 \leq x \leq 10$ ,  $-10 \leq y \leq 10$  is given by the function

$$T(x, y) = 100 - x^2 - y^2.$$

- (a) Sketch isothermal curves (curves of constant temperature) for  $T = 100^{\circ}\text{C}$ ,  $T = 75^{\circ}\text{C}$ ,  $T = 50^{\circ}\text{C}$ ,  $T = 25^{\circ}\text{C}$ , and  $T = 0^{\circ}\text{C}$ .
- (b) A heat-seeking bug is put down at a point on the  $xy$ -plane. In which direction should it move to increase its temperature fastest? How is that direction related to the level curve through that point?
33. Use the factored form of  $f(x, y) = x^2 - y^2 = (x - y)(x + y)$  to sketch the contour  $f(x, y) = 0$  and to find the regions in the  $xy$ -plane where  $f(x, y) > 0$  and the regions where  $f(x, y) < 0$ . Explain how this sketch shows that the graph of  $f(x, y)$  is saddle-shaped at the origin.
34. Use Problem 33 to find a formula for a “monkey-saddle” surface  $z = g(x, y)$  which has three regions with  $g(x, y) > 0$  and three with  $g(x, y) < 0$ .
35. Use the contour diagram for  $f(x, t) = \cos t \sin x$  in Figure 12.61 to describe in words the cross-sections of  $f$

with  $t$  fixed and the cross-sections of  $f$  with  $x$  fixed. Explain what you see in terms of the behavior of the string.

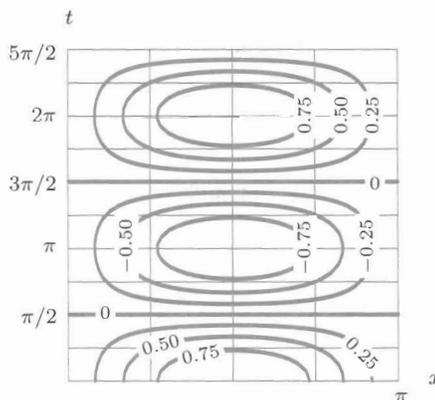


Figure 12.61

## 12.4 LINEAR FUNCTIONS

### What is a Linear Function of Two Variables?

Linear functions played a central role in one-variable calculus because many one-variable functions have graphs that look like a line when we zoom in. In two-variable calculus, a *linear function* is one whose graph is a plane. In Chapter 14, we see that many two-variable functions have graphs which look like planes when we zoom in.

#### What Makes a Plane Flat?

What makes the graph of the function  $z = f(x, y)$  a plane? Linear functions of *one* variable have straight line graphs because they have constant slope. On a plane, the situation is a bit more complicated. If we walk around on a tilted plane, the slope is not always the same: it depends on the direction in which we walk. However, at every point on the plane, the slope is the same as long as we choose the same direction. If we walk parallel to the  $x$ -axis, we always find ourselves walking up or down with the same slope; the same is true if we walk parallel to the  $y$ -axis. In other words, the slope ratios  $\Delta z / \Delta x$  (with  $y$  fixed) and  $\Delta z / \Delta y$  (with  $x$  fixed) are each constant.

**Example 1** A plane cuts the  $z$ -axis at  $z = 5$ , has slope 2 in the  $x$  direction and slope  $-1$  in the  $y$  direction. What is the equation of the plane?

**Solution** Finding the equation of the plane means constructing a formula for the  $z$ -coordinate of the point on the plane directly above the point  $(x, y)$  in the  $xy$ -plane. To get to that point start from the point above the origin, where  $z = 5$ . Then walk  $x$  units in the  $x$  direction. Since the slope in the  $x$  direction is 2, the height increases by  $2x$ . Then walk  $y$  units in the  $y$  direction; since the slope in the  $y$  direction is  $-1$ , the height decreases by  $y$  units. Since the height has changed by  $2x - y$  units, the  $z$ -coordinate is  $5 + 2x - y$ . Thus, the equation for the plane is

$$z = 5 + 2x - y.$$