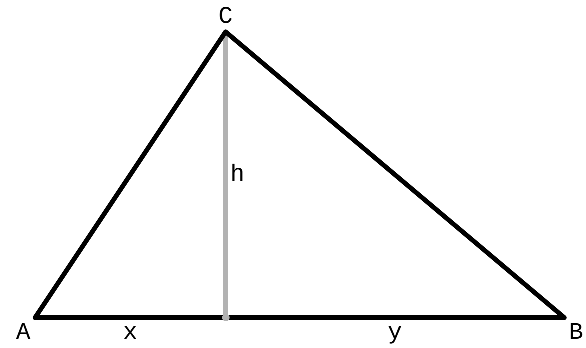
Problem. Prove that among all triangles with a fixed area the equilateral triangle has the smallest perimeter.

Solution. We will use the variables x, y and h as indicated in the figure below. With the variables x, y and h the area A and the perimeter P are given as follows:

$$A = \frac{1}{2}(x+y)h$$

$$P = x + y + \sqrt{x^2+h^2} + \sqrt{y^2+h^2}$$



Applying the method of Lagrange multipliers we get the equations

$$(x+y)h = 2A$$

$$1 + \frac{x}{\sqrt{x^2+h^2}} = \lambda h \qquad (1)$$

$$1 + \frac{y}{\sqrt{y^2+h^2}} = \lambda h \qquad (2)$$

$$\frac{h}{\sqrt{x^2+h^2}} + \frac{h}{\sqrt{y^2+h^2}} = \lambda(x+y)$$

Equations (1) and (2) yield $x = y$. So the above four equations reduce to the following three equations

$$xh = A \qquad (3)$$

$$1 + \frac{x}{\sqrt{x^2+h^2}} = \lambda h \qquad (4)$$

$$\frac{h}{\sqrt{x^2+h^2}} = \lambda x \qquad (5)$$

Multiplying (4) by x and (5) by h we obtain

$$x + \frac{x^2}{\sqrt{x^2+h^2}} = \frac{h^2}{\sqrt{x^2+h^2}},$$

or, equivalently,

$$x = \frac{h^2 - x^2}{\sqrt{x^2+h^2}}.$$

Squaring the last equation and simplifying we get

$$x^2\left(x^2+h^2\right) = \left(h^2-x^2\right)^2.$$

Expending both sides we get

$$x^4 + (xh)^2 = h^4 - 2(hx)^2 + x^4.$$

Now we use (3) and further simplify

$$A^2 = h^4 - 2A^2.$$

Hence

$$h = \sqrt[4]{3}\,\sqrt{A}, \qquad \text{and} \qquad x = \frac{A}{h} = \frac{1}{\sqrt[4]{3}}\,\sqrt{A}.$$

Now we calculate the sides of the triangle:

$$\overline{AB} = x + y = \frac{2}{\sqrt[4]{3}}\,\sqrt{A}$$

$$\overline{BC} = \sqrt{y^2 + h^2} = \sqrt{\frac{1}{\sqrt{3}}\,A + \sqrt{3}\,A} = \sqrt{\frac{1}{\sqrt{3}}\,A + \frac{3}{\sqrt{3}}\,A} = \frac{2}{\sqrt[4]{3}}\,\sqrt{A}$$

$$\overline{CA} = \sqrt{x^2 + h^2} = \sqrt{\frac{1}{\sqrt{3}}\,A + \sqrt{3}\,A} = \sqrt{\frac{1}{\sqrt{3}}\,A + \frac{3}{\sqrt{3}}\,A} = \frac{2}{\sqrt[4]{3}}\,\sqrt{A}$$

Thus, the triangle obtained by the Lagrange method is equilateral.
Since the Lagrange method produced only one possible extreme value for the perimeter and since for a small h the corresponding P is large, we conclude that we obtained a triangle with the minimal perimeter.
