Section 12.1 Functions of two variables

Key concepts:

- The coordinate system in 3-space
- Graphing equations in 3-space
- Distance between two points $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ is given by

$$d(P_1, P_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

Section 12.1, Exercises and Problems: 1 - 7, 22, 23, 24, 27, 28, 29, 30, 31, 32, 33, 34

Problem. Find eight vertexes of the cube with the following three properties:

- The cube contains the sphere $(x-1)^2 + (y-2)^2 + (z-3)^2 = 16$.
- Each face of the cube is parallel to a coordinate plane.
- Each face of the cube touches the sphere.

Find the points where the faces touch the sphere.

Section 12.2 Graphs of functions of two variables

In this section we study graphs of functions of two variables x and y. The graph of a function $z = f(x, y)$ is a surface in the xyz-space. The domain a function $z = f(x, y)$ is a set in xy-plane. Most of the functions in this section are defined on the whole xy-plane. However, one should pay attention to the exceptions. For example, the domains of the following functions are subsets of the xy-plane:

$$\begin{aligned} z = f_1(x, y) &= \frac{1}{x^2 + y^2}, && \text{is defined for all } (x, y) \text{ such that } (x, y) \neq (0, 0); \\ z = f_2(x, y) &= \sqrt{1 - x^2 - y^2}, && \text{is defined for all } (x, y) \text{ such that } x^2 + y^2 \leq 1; \\ z = f_3(x, y) &= \sqrt{x^2 + y^2 - 1}, && \text{is defined for all } (x, y) \text{ such that } x^2 + y^2 \geq 1. \end{aligned}$$

Section 12.2 Exercises and Problems: 1 - 18, 20, 22, 23, 27, 28

Section 12.3 Contour diagrams

In this section we study contours. The contour of a function $z = f(x, y)$ at height c is the set of all points (x, y) such that $f(x, y) = c$. Here f is a given formula and c is a number that we choose. To identify a contour in xy-plane we need to solve the equation $f(x, y) = c$ for x and y. It is important to identify those numbers c for which the equation $f(x, y) = c$ does not have a solution. Often such numbers c are not difficult to find. For example

$$\begin{aligned} \frac{1}{x^2 + y^2} &= c, && \text{does not have a solution for } c < 0; \\ \sqrt{1 - x^2 - y^2} &= c, && \text{does not have a solution for } c < 0 \text{ or } c > 1; \\ \sqrt{x^2 + y^2 - 1} &= c, && \text{does not have a solution for } c < 0. \end{aligned}$$

> The set of all numbers c for which the equation $f(x, y) = c$ has a solution is called the **range** of f.

Section 12.3, Exercises and Problems: 5 - 16, 22, 24, 27 - 30, 32, 33, 34

Problem. Find the ranges of the functions f_1, f_2, f_3 defined above.

Problem. Find the ranges of the functions in #4, #22(d), #28.

Section 12.4 Linear Functions

Key concepts:

- Let k, m and n be given numbers. The graph of the function $f(x,y) = k + mx + ny$ in xyz-space is a plane. This plane has slope m in the x direction and slope n in the y direction. Its z-intercept is the point $(0,0,k)$. Its contour lines (as lines in xy-plane) have slope $-m/n$.
- If a plane has slope m in the x direction and slope n in the y direction, and passes through the point (x_0, y_0, z_0), then its equation is

$$z = z_0 + m(x - x_0) + n(y - y_0).$$

Section 12.4 Exercises and Problems: 1 - 14, 18, 21 - 23, 25, 27, 29

Section 12.5 Functions of Three Variables

Key concepts:

- A level surface of a function $g(x,y,z)$ of three variables is the set of all points (x,y,z) such that $g(x,y,z) = c$, where c is a constant. For example, if $g(x,y,z) = x^2 + y^2 - z$, then the surface represented by $x^2 + y^2 - z = 0$ is a level surface (at the level 0) of g. Solving for z we get $z = x^2 + y^2$. This surface is called a paraboloid. Another level surface (at the level 1) is $x^2 + y^2 - z = 1$. Solving for z we get $z = -1 + x^2 + y^2$. This is also a paraboloid; it is a a vertical translation of the previous paraboloid by -1.
- A family of level surfaces is used to represent a function of three variables. For example, the function $g(x,y,z) = x^2 + y^2 - z$ is represented by a family of paraboloids $x^2 + y^2 - z = c$.
- A single surface is a graph of a two variable function. For example, if $f(x,y) = x^2 + y^2$, then the surface $z = x^2 + y^2$ is a the graph of f. This same surface can be thought of as one member of the family of level surfaces of the three-variable function $g(x,y,z) = x^2 + y^2 - z$.
- Study and learn the surfaces in the catalog on page 636. The case $a = b = c = 1$ is the most significant case.

Section 12.5 Exercises and Problems: 1 - 13 (odd), 20, 21, 22, 23, 24, 26, 27, 28, 29, 31

Section 12.6 Limits and Continuity

- Informally, a function $f(x,y)$ of two variables is continuous at a point (a,b) if the values $f(x,y)$ do not differ much from $f(a,b)$ whenever the distance of (x,y) to (a,b) is small. In other words, we can make $f(x,y)$ as close as we wish to $f(a,b)$ by taking (x,y) close to (a,b).
- As a rule of thumb we can say that any function given by a relatively simple formula is continuous at each point where it is defined. The domain should be easy to recognize: division by 0 is not allowed, square roots of negative numbers are not defined, ln is defined only for positive numbers, etc.
- Understand why the function $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$ is not continuous at $(0,0)$.

Section 12.6 Exercises and Problems: 1 - 11, 13, 15, 19, 20, 21, 23

Review Exercises and Problems for Ch. 12: 7, 15, 16, 21, 28, 29, 36 - 39