Chapter 15 Review Problem 46. An irrigation canal has trapezoidal cross section of area A. Minimize the perimeter p.

Solution. In the book they give three variables: d, w and θ. Then the area A is given by

$$A = \left(w + \frac{d}{\tan\theta}\right) d.$$

The perimeter p to be minimized is

$$p = w + 2\,\frac{d}{\sin\theta}.$$

Solving the the first equation for w and substituting in the formula for the perimeter we get

$$p = \frac{A}{d} - \frac{d}{\tan\theta} + 2\,\frac{d}{\sin\theta}.$$

The last expression simplifies to

$$p = \frac{A}{d} + d\,\frac{2-\cos\theta}{\sin\theta}.$$

To find the critical points of p as a function of d and θ we find

$$\frac{\partial p}{\partial d} = -\frac{A}{d^2} + \frac{2-\cos\theta}{\sin\theta} \quad \text{and} \quad \frac{\partial p}{\partial \theta} = d\,\frac{(\sin\theta)^2 - (2-\cos\theta)\cos\theta}{(\sin\theta)^2}.$$

To find the critical points we simplify the expression for $\dfrac{\partial p}{\partial \theta}$ and solve the equations

$$-\frac{A}{d^2} + \frac{2-\cos\theta}{\sin\theta} = 0, \qquad d\,\frac{1-2\cos\theta}{(\sin\theta)^2} = 0.$$

Since by the nature of our problem we have $d > 0$ and $0 < \theta < \pi/2$, the second equation is equivalent to $1 - 2\cos\theta = 0$. The solution of this equation is $\theta = \pi/3$. Substituting this value in the first equation we get

$$-\frac{A}{d^2} + \frac{2-1/2}{\sqrt{3}/2} = 0.$$

Solving for d we get $d = \sqrt{A/\sqrt{3}}$. Now calculate the value of p for $\theta = \pi/3$ and $d = \sqrt{A/\sqrt{3}}$:

$$p = \sqrt{A\sqrt{3}} + \sqrt{\frac{A}{\sqrt{3}}}\,\frac{2-1/2}{\sqrt{3}/2} = 2\sqrt{A\sqrt{3}}.$$

It is also interesting to calculate the corresponding w:

$$w = \frac{A}{\sqrt{A/\sqrt{3}}} - \frac{\sqrt{A/\sqrt{3}}}{\sqrt{3}} = \sqrt{A\sqrt{3}} - \sqrt{\frac{A}{3\sqrt{3}}} = \sqrt{A\sqrt{3}} - \sqrt{\frac{A\sqrt{3}}{3\cdot 3}} = \sqrt{A\sqrt{3}}\left(1 - \frac{1}{3}\right) = \frac{2}{3}\sqrt{A\sqrt{3}}.$$

Thus the minimum perimeter is three lengths of w. This means that the optimal dimensions of an irrigation canal leads to w equaling the slanted sides of the canal.

It remains to check whether the critical point that we found is really a global minimum. We calculate

$$\frac{\partial^2 p}{\partial d^2} = 2\frac{A}{d^3}$$
$$\frac{\partial^2 p}{\partial \theta \partial d} = \frac{1 - 2\cos\theta}{(\sin\theta)^2}$$
$$\frac{\partial p^2}{\partial \theta^2} = d\,\frac{2(\sin\theta)^3 - 2(1 - 2\cos\theta)\sin\theta}{(\sin\theta)^4}.$$

Next we substitute the values $\theta = \pi/3$ and $d = \sqrt{A/\sqrt{3}}$ found for the critical point:

$$\frac{\partial^2 p}{\partial d^2} = 2\frac{3^{3/4}}{\sqrt{A}}$$
$$\frac{\partial^2 p}{\partial \theta \partial d} = 0$$
$$\frac{\partial p^2}{\partial \theta^2} = \frac{4\sqrt{A}}{3^{3/4}}.$$

Hence $D > 0$ and since $\frac{\partial^2 p}{\partial d^2} > 0$ we conclude that the critical point is really a local minimum.

It is clear from the formula for the perimeter that for each fixed $0 < \theta < \pi/2$ for d close to 0 the perimeter is huge. Also for each fixed $0 < \theta < \pi/2$ for a huge p the perimeter is also huge. This tells us that the local minimum that we found is in fact a global minimum. That is also confirmed with the following contour plots of p. In the plot below I chose $A = 1$. Figure 1 shows the contours which are 0.1 units apart. Figure 1 shows the magnified view near the critical point. The contours are 0.025 units apart.

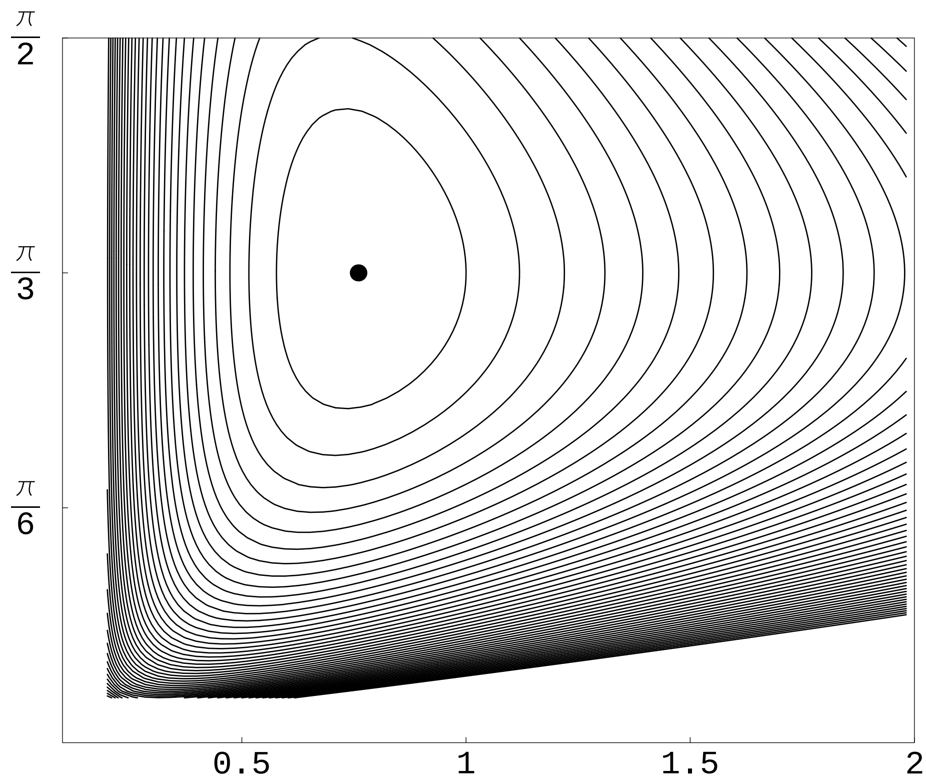

Figure 1: A contour plot for p, contours are 0.1 units apart

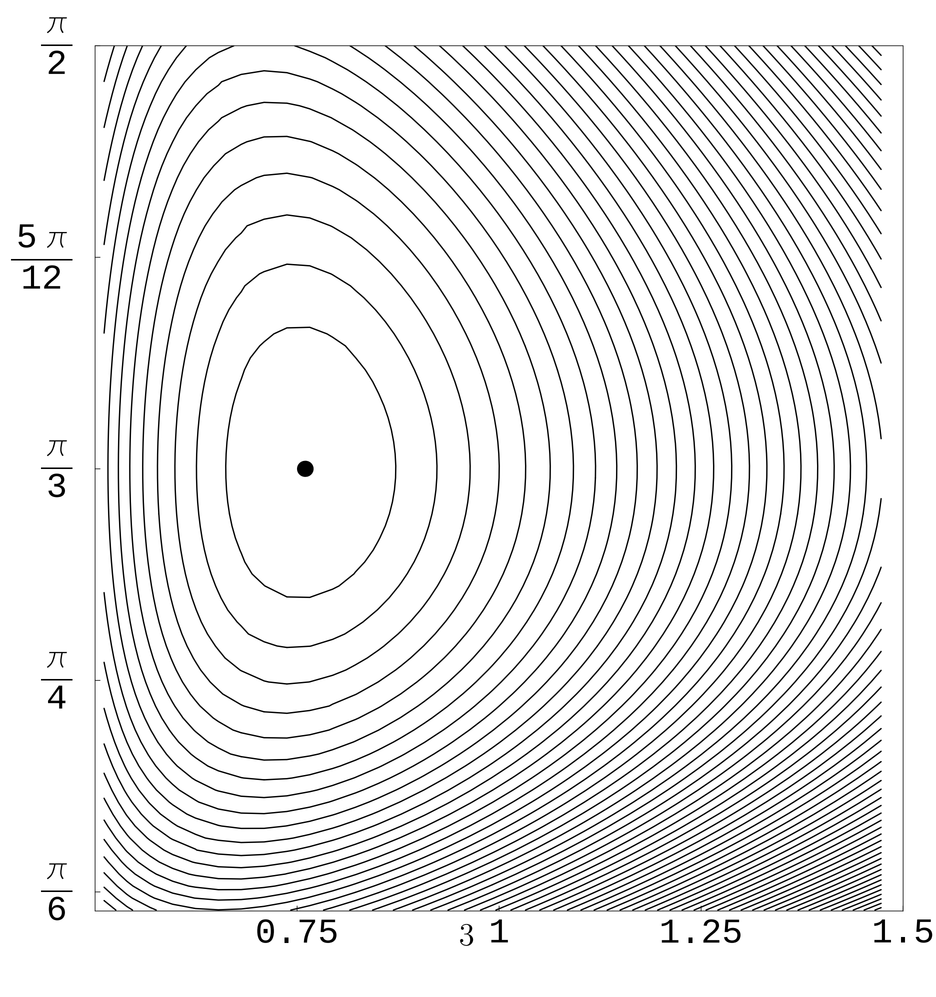

Figure 2: A contour plot for p, contours are 0.025 units apart