Section 14.1 Partial derivative

- Pay attention to the notation: $f_x(x,y)$, $f_y(x,y)$, $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial f}{\partial y}$ and several other forms
- Visualizing partial derivatives on a graph
- Estimating partial derivatives using difference quotients
- Estimating partial derivatives from a contour diagram
- Using units to interpret partial derivatives

Section 14.1, Exercises and Problems: 1, 2, 4, 5, 6, 8, 9 - 12, 16, 17, 20 - 23, 30

Section 14.2 Computing partial derivatives algebraically

- Review all derivative rules from Math 124
- What partial derivatives mean in practical terms?

Section 14.2, Exercises and Problems: 1 - 35 (do as many as you can), 36, 37, 38, 40, 42, 43, 44, 45

Section 14.3 Local linearity and the differential

- Let $z = f(x,y)$ be a function of two variables. Let (a,b) be a point in the domain of f. Then the point $\big(a,b,f(a,b)\big)$ is on the graph of f. The graph of f is the surface $z = f(x,y)$. The equation of the tangent plane to the surface $z = f(x,y)$ at the point $\big(a,b,f(a,b)\big)$ is

$$\boxed{z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)}$$

- The tangent plane approximation to $f(x,y)$ near (a,b) - the local linearization
- The differential

Section 14.3, Exercises and Problems: 1 - 12, 17, 18, 20, 21, 22, 23, 24, 30

Section 14.4 Gradients and the directional derivative in the plane

Let $z = f(x,y)$ be a differentiable function of two variables. Let (a,b) be a point in the domain of f.

- The **gradient vector** of f at the point (a,b) is

$$\boxed{\big(\overrightarrow{\text{grad}}\, f\big)(a,b) = \big(\overrightarrow{\nabla} f\big)(a,b) = f_x(a,b)\,\overrightarrow{i} + f_y(a,b)\,\overrightarrow{j}}$$

- The **directional derivative** of f at the point (a,b) in the direction of the unit vector $\overrightarrow{u} = u_1\,\overrightarrow{i} + u_2\,\overrightarrow{j}$ is

$$\boxed{f_{\overrightarrow{u}}(a,b) = \Big(\big(\overrightarrow{\nabla} f\big)(a,b)\Big)\cdot\overrightarrow{u} = f_x(a,b)u_1 + f_y(a,b)u_2}$$

Section 14.4, Exercises and Problems: 1 - 43 (do as many as you can), 46, 48, 53 - 57, 63, 65 - 70, 73, 76, 78, 80

Section 14.5 Gradients and the directional derivative in space

Let $w = f(x,y,z)$ be a differentiable function of three variables. Let (a,b,c) be a point in the domain of f.

- The **gradient vector** of f at the point (a,b,c) is

$$\big(\overrightarrow{\text{grad}}\, f\big)(a,b,c) = \big(\overrightarrow{\nabla} f\big)(a,b,c) = f_x(a,b,c)\,\overrightarrow{i} + f_y(a,b,c)\,\overrightarrow{j} + f_z(a,b,c)\,\overrightarrow{k}$$

- The **directional derivative** of f at the point (a,b,c) in the direction of the unit vector $\overrightarrow{u} = u_1\overrightarrow{i} + u_2\overrightarrow{j} + u_3\overrightarrow{k}$ is

$$f_{\overrightarrow{u}}(a,b,c) = \big(\big(\overrightarrow{\nabla} f\big)(a,b,c)\big) \cdot \overrightarrow{u} = f_x(a,b,c)u_1 + f_y(a,b,c)u_2 + f_z(a,b,c)u_3$$

Section 14.5, Exercises and Problems: 1 - 40 (do as many as you can; choose various types), 41, 42, 44, 46, 48, 49, 51, 54, 55, 58.

Section 14.6 The chain rule I think that the book makes too big of a deal of formulas presented here. The main formula is the boxed formula on page 760. Below I write the boxed formula on page 760 in a more detailed form. Let $F(x,y)$ be a function of two variables and $g(t)$ and $h(t)$ functions of one variable. Then $k(t) = F\big(g(t), h(t)\big)$ is a composite function of one variable. The derivative of the function k is given by

$$k'(t) = F_x\big(g(t), h(t)\big)g'(t) + F_y\big(g(t), h(t)\big)h'(t)$$

The boxed formula on page 762 is follows the same structure. Below I write the boxed formula on page 762 in the spirit of the formula above. Let $F(x,y)$ be a function of two variables. Instead of $g(t)$ consider a function of two variables $G(s,t)$ and instead of $h(t)$ consider a function of two variable $H(s,t)$. Then $K(s,t) = F\big(G(s,t), H(s,t)\big)$ is a composite function of two variables. However, when we calculate $K_t(s,t)$, the partial derivative of K with respect to t we consider s to be a constant, so we essentially have the same situation as above:

$$K_t(s,t) = F_x\big(G(s,t), H(s,t)\big)G_t(s,t) + F_y\big(G(s,t), H(s,t)\big)H_t(s,t)$$

To calculate $K_s(s,t)$, the partial derivative of K with respect to s, we consider t to be a constant. We have

$$K_s(s,t) = F_x\big(G(s,t), H(s,t)\big)G_s(s,t) + F_y\big(G(s,t), H(s,t)\big)H_s(s,t)$$

Please recognize that the three formulas above follow the same logic.

Next, I present two formulas which are not in the book. These formulas are just a two variable versions of the chain rule from Math 124. Here is the chain rule from Math 124. Let $f(x)$ be a function of one variable and let $g(t)$ also be a function of one variable. Then $k(t) = f\big(g(t)\big)$ is a composite function of one variable. The derivative of the function k is given by

$$k'(t) = f'\big(g(t)\big)g'(t)$$

Now replace the function $g(t)$ with a function of two variables $G(s,t)$. Then $K(s,t) = f\big(G(s,t)\big)$ is a composite function of two variables. The partial derivatives of $K(s,t)$ are given by

$$\begin{aligned} K_s(s,t) &= f'\big(G(s,t)\big)G_s(s,t) \\ K_t(s,t) &= f'\big(G(s,t)\big)G_t(s,t) \end{aligned}$$

These formula are important, for example in problem 20.

Section 14.6, Exercises and Problems: 1 - 14, 16, 18, 20, 27, 29, 30, 31

Section 14.7 Second order partial derivatives

The first objective here is to understand the meaning of the second partial derivatives of a function of two variables. One test for that is the equation for an oscillating string, see the class website for an animation.

The most important formula here is the quadratic approximation for a function of two variables shown in the box below. For a function $F(x,y)$ which has continuous second-order partial derivatives the following approximation holds:

$$\begin{aligned} F(x,y) \approx & F(a,b) + F_x(a,b)\,(x-a) + F_y(a,b)\,(y-b) \\ & + \tfrac{1}{2}F_{xx}\,(a,b)(x-a)^2 + F_{xy}(a,b)\,(x-a)(y-b) + \tfrac{1}{2}F_{yy}(a,b)\,(y-b)^2 \end{aligned}$$

Notice that the first part of the approximation (which is on the first line of the boxed formula) is the linear approximation of $F(x,y)$ near (a,b).

Section 14.7, Exercises and Problems: 1 - 30 (do most), 32, 33, 35, 36, 40, 41, 44, 45.

Chapter 14, Review Exercises and Problems: 45, 48, 49, 54, 55, 57, 58, 59, 102