MATH 224 Examination 3 February 17, 2012 Name ____________________

Give detailed explanations for your answers. There are four problems. Each is worth 25 points.

1. Figure 1 shows an oscillating string. The equation of the oscillating string is $y = F(x,t) = (\sin x)(\cos t)$ where $x \in [0,\pi)$, $t \geq 0$. Here, for a fixed time $t = t_0$, $y = F(x, t_0)$ describes the shape of the string at time t_0. In Figure 1 the string at time $t = t_0$ is black. To indicate the motion of the string, I added several previous positions of the string in various shade of gray. Consider the following seven quantities:

$$F(x_0,t_0),\ F_x(x_0,t_0),\ F_t(x_0,t_0),\ F_{xx}(x_0,t_0),\ F_{xt}(x_0,t_0),\ F_{tx}(x_0,t_0),\ F_{tt}(x_0,t_0).$$

Based on Figure 1 for each of the seven quantities listed above determine whether it is positive or negative.

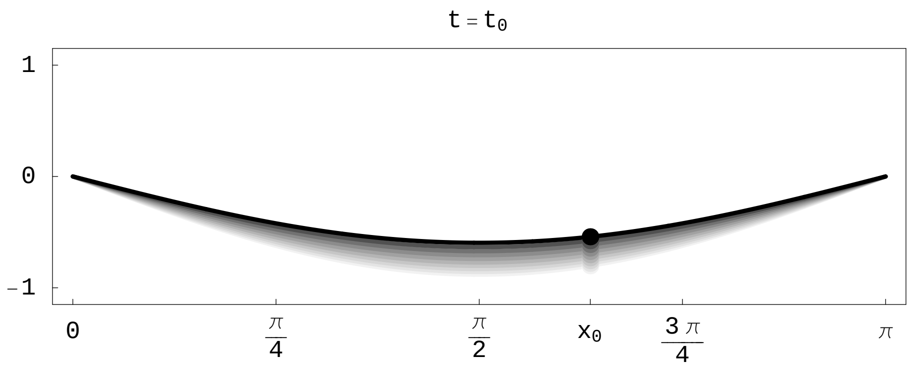


Figure 1: An oscillating string

2. The goal of this problem is to make the cheapest storage box with a fixed volume, as shown in Figure 2. For simplicity we can assume that the fixed volume is 1 cubic unit. As you can see in Figure 2 the storage box is build on a side of a house. It has three vertical "walls" made of chain-link fencing and the roof. The roofing material costs three times as much (per square unit) as chain-link fencing. Find the dimensions (depth, width and height) of the storage box that will minimize the cost of the materials.
Give both: exact and approximate values for the dimensions of the box.
Use the **second derivative test** to confirm that the point you obtained is a local minimum.

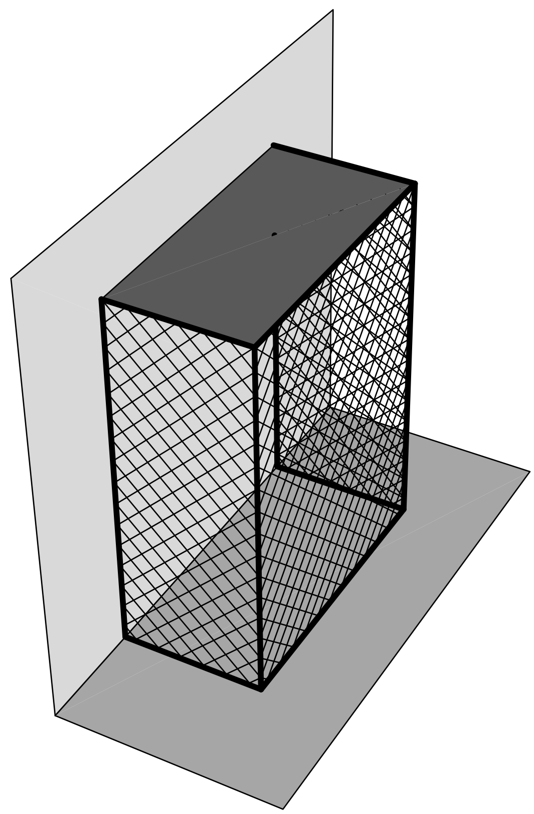


Figure 2: A storage

3. Consider the function $F(x,y) = 4x\sqrt{y} - 4\ln(xy)$. (You can think of F as being a temperature at each point of a heated plate.) Consider the point $P = (4,1)$.
   (a) Find the vector in the direction of maximum rate of change of F at the point P. What is the maximum rate of change of F?
   (b) Find the instantaneous rate of change of F as you leave P heading toward the point $(2,3)$.
   (c) Find a vector in a direction in which the rate of change of F at P is 0.

4. Consider the hyperboloid $x^2 + y^2 - z^2 = 1$. Is there a point on this hyperboloid at which the tangent plane to the hyperboloid is parallel to the plane $x + y + z = 0$? If so, find it, if not explain why not. If there is more than one such point find all of them.

1. $F(x_0,t_0)<0$ position is below x-axis $\boxed{1}$

$F_x(x_0,t_0)>0$ the slope of the string is >0

$F_t(x_0,t_0)>0$ ~~the~~ string's position is <u>increasing</u>

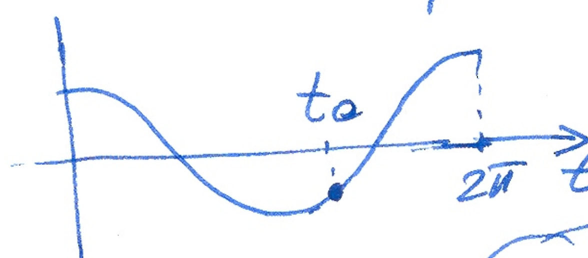
$F_{xx}(x_0,t_0)>0$ the string is smiling.

$F_{xt}(x_0,t_0)<0$ the slope is decreas. in time

$F_{tx}(x_0,t_0)<0$ the velocity is ~~increasing~~ decreasing with increasing x

$F_{tt}(x_0,t_0)>0$ the string is speeding up.

plot cost



2. $3xy+2xz+yz \rightarrow$ is the cost of material

$xyz=1$ volume

$z=\frac{1}{xy}$

$C(x,y)=3xy+\frac{2}{y}+\frac{1}{x}$

② Find CP-s :

$$C_x = 3y - \frac{1}{x^2} = 0$$

$$C_y = 3x - \frac{2}{y^2} = 0$$

solve for $x, y > 0$.

$$y = \frac{1}{3x^2}, \quad 3x - \frac{2}{\frac{1}{9x^4}} = 0$$

$$3x - 18x^4 = 0$$

$$6x^3 = 1, \quad x = 1/\sqrt[3]{6}$$

$$y = \frac{1}{3\,6^{-2/3}} = \frac{6^{2/3}}{3} = \frac{2^{2/3}\,3^{2/3}}{3}$$

$$= \frac{2^{2/3}}{3^{1/3}} = \sqrt[3]{4/3}$$

$$z = \frac{1}{\cancel{\frac{1}{\sqrt[3]{6}}}\,xy} = \frac{1}{\cancel{x}\,\frac{1}{3x^{2}}} = \underline{\underline{3x}} = \underline{\frac{3}{\sqrt[3]{6}}}$$

---

the second derivative test

$$C_{xx} = 2/x^3 \qquad C_{xy} = 3 \qquad C_{yy} = 4/y^3$$

$$D = 8/(xy)^3 - 9 = 8\cdot\cancel{27}\,9 - 9 = 36 - 9 > 0$$

(3)

$$F_x = 4\sqrt{y} - \frac{4}{x} \qquad F_y = \frac{2x}{\sqrt{y}} - \frac{4}{y}$$

8 −

at (4,1)

3

4

$\boxed{3}$

(a)

$$(\vec{\nabla}F)(4,1) = \underline{3\vec{i} + 4\vec{j}}$$

direction of max change

$$\|(\vec{\nabla}F)(4,1)\| = \underbrace{5}_{\text{max rate of chang.}}$$

(b)

$$2\vec{i} + 3\vec{j} - (4\vec{i} + \vec{j}) = -2\vec{i} + 2\vec{j}$$

$$\vec{u} = \frac{1}{\sqrt{2}}(-\vec{i} + \vec{j})$$

$$\vec{u} \cdot \vec{\nabla}F = -\frac{3}{\sqrt{2}} + \frac{4}{\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

(c) One such vector is $-4\vec{i} + 3\vec{j}$ or another one $4\vec{i} - 3\vec{j}$, orthogonal to $(\vec{\nabla}F)(4,1)$.

④ $\vec{n} = \vec{i} + \vec{j} + \vec{k}$ $\boxed{14}$

set $H(x,y,z) = x^2+y^2-z^2$

$(\vec{\nabla} H)(x,y,z) = 2x\vec{i} + 2y\vec{j} - 2z\vec{k}$

Is it possible to find λ such that

$2x\vec{i} + 2y\vec{j} - 2z\vec{k} = \lambda(\vec{i}+\vec{j}+\vec{k})$, or

$$\left.\begin{array}{lll} 2x = \lambda & \Rightarrow & x = \frac{\lambda}{2} \\ 2y = \lambda & & y = \frac{\lambda}{2} \\ -2z = \lambda & & z = -\frac{\lambda}{2} \end{array}\right\}$$ we need $x^2+y^2-z^2 = 1$ so

$$\frac{\lambda^2}{4} + \cancel{\frac{\lambda^2}{4}} - \cancel{\frac{\lambda^2}{4}} = 1$$

Thus $\lambda^2 = 4$ or $\lambda = 2$ or $\lambda = -2$

This gives us two points

$(1, 1, -1)$ and $(-1, -1, 1)$.

At these two points tangent plane to Hyperboloid are $\parallel$ to the given plane.