MATH 226 Examination 1
April 28, 2009

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Give all details of your reasoning. Each problem is worth 25 points for the total of 100 points.

Problem 1. (a) Write without the absolute values the exact value of the expression

$$\left|\pi^e - e^\pi\right| .$$

(b) Write the following English sentence as an inequality involving absolute value:

$$\boxed{\text{The distance between a number } x \text{ and the number } -\frac{2}{3} \text{ is less than } \frac{1}{4}.}$$

Illustrate with a diagram on the number line.

Problem 2. (a) State the definition of the absolute value function.

(b) State all the properties of absolute value that you will need in (c). (No proofs are required, just the statements. You can not list any version of the triangle inequality here.)

(c) Prove that $\left|a+b\right| \leq \left|a\right| + \left|b\right|$ for all $a, b \in \mathbb{R}$.

Problem 3. (a) State the definition of

$$\lim_{x \to +\infty} f(x) = L .$$

(b) Use the definition of limit to prove that

$$\lim_{x \to +\infty} \frac{x}{x + \cos x} = ? .$$

Problem 4. (a) State the ϵ-δ definition of continuity of a function f at a point a.

(b) Use ϵ-δ definition of continuity to prove that the function

$$f(x) = \frac{1}{x^2}$$

is continuous on $(0, +\infty)$.

Megan Boddy
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① ⓐ $|\pi^e - e^\pi|$ is distance between π^e and e^π

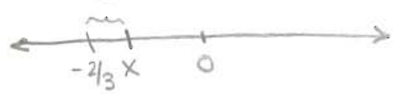
$\pi^e \approx 22.46$

$e^\pi \approx 23.14$

$e^\pi > \pi^e$ so $|\pi^e - e^\pi| = \boxed{e^\pi - \pi^e}$

ⓑ $|x - (\frac{-2}{3})| < \frac{1}{4}$

this distance is less than $\frac{1}{4}$



* x could be to the left or to the right of $-2/3$

② a) $|x| = \max\{x, -x\}$

b) for $x \geq 0$ $|x| = x$

$x < 0$ $|x| = -x$

$$\boxed{\begin{array}{l} |x| \geq x \\ |x| \geq -x \end{array}}$$

c) prove $|a+b| \leq |a| + |b|$

$|a+b| = \max\{a+b, -a-b\}$

I need to show ① $a+b \leq |a| + |b|$

② $-a-b \leq |a| + |b|$

① $|a| \geq a$ and $|b| \geq b$

so $|a| + |b| \geq a + b$

② $|a| \geq -a$ and $|b| \geq -b$

so $|a| + |b| \geq -a + -b$

since $|a| + |b| \geq a+b$ and $|a| + |b| \geq -(a+b)$

$|a| + |b| \geq \max\{a+b, -(a+b)\}$

by definition of absolute value:

$|a| + |b| \geq |a+b|$

3 a $\lim_{x\to+\infty} f(x) = L$

I) $\exists X_0$ s.t. $f(x)$ is defined for all $x > X_0$

II) $\forall \varepsilon > 0 \ \exists X(\varepsilon) > X_0$ s.t. $x > X(\varepsilon) \Rightarrow |f(x) - L| < \varepsilon$

b $\lim_{x\to\infty} \frac{x}{x+\cos x} = 1$

I) problems with $f(x)$ being defined happen when $x + \cos x = 0$ if $X_0 = 2$ then $x+\cos x$ is always greater than zero. let $\boxed{X_0 = 2}$

II) find $X(\varepsilon)$:

$\left|\frac{x}{x+\cos x} - 1\right| < \varepsilon$ solve for x

$\left|\frac{x}{x+\cos x} - 1\right| = \left|\frac{x - x - \cos x}{x + \cos x}\right| = \left|\frac{-\cos x}{x+\cos x}\right| = \frac{|-\cos x|}{|x+\cos x|}$

Now play pizza party:

$\frac{\overbrace{|-\cos x|}^{\text{biggest}=1}}{\underbrace{|x+\cos x|}_{\text{smallest}=-1}} \le \frac{1}{\underbrace{|x-1|}_{\text{for } x>2,\ |x-1|>0}} = \frac{1}{x-1}$

$\boxed{\text{BIN}}$ for $x > 2$, $\left|\frac{x}{x+\cos x} - 1\right| \le \frac{1}{x-1}$

solve $\frac{1}{x-1} < \varepsilon$ for x:

$\frac{1}{\varepsilon} < x - 1 \rightarrow \boxed{\frac{1}{\varepsilon} + 1 < x}$ and $\underline{x > 2}$

$\boxed{\text{So } X(\varepsilon) = \max\left\{\frac{1}{\varepsilon} + 1,\ 2\right\}}$

Prove: $x > X(\varepsilon) \Rightarrow |f(x) - 1| < \varepsilon$

$x > 2$ and $x > \frac{1}{\varepsilon} + 1$

This means I can use BIN

$x - 1 > \frac{1}{\varepsilon}$

$\varepsilon > \frac{1}{x-1}$

BIN Tells me $\frac{1}{x-1} \ge \left|\frac{x}{x+\cos x} - 1\right|$

if $\varepsilon > \frac{1}{x+1}$ then

$\boxed{\varepsilon > \left|\frac{x}{x+\cos x} - 1\right|}$

4) a) $f(x)$ is continuous at a if:

including 0 – $f(x)$ must be defined at a

I) $\exists\ \delta_0 > 0$ s.t. $f(x)$ is defined for all $|x-a| \leq \delta_0$

II) $\forall\ \varepsilon > 0\ \exists\ \delta(\varepsilon)$ s.t. $0 < \delta(\varepsilon) < \delta_0$ and $|x-a| < \delta(\varepsilon) \Rightarrow$ $|f(x) - f(a)| < \varepsilon$

b) $f(x) = \frac{1}{x^2}$ continuous on $(0, +\infty)$

I) problem with $f(x)$ being defined is at $x=0$. I'll let $\delta_0 = \frac{a}{2}$ so that $|x-a| < \frac{a}{2}$ (my world: $x \in (\frac{a}{2}, \frac{3a}{2})$)

II) solve $|f(x) - f(a)| < \varepsilon$ for $|x-a|$

$$\left| \frac{1}{x^2} - \frac{1}{a^2} \right| = \left| \frac{a^2 - x^2}{x^2 a^2} \right| = \left| \frac{(a-x)(a+x)}{x^2 a^2} \right| = \frac{|-(x-a)|\,|x+a|}{x^2 a^2}$$

$$= \frac{|x-a|\,|x+a|}{x^2 a^2} \leq \frac{|x-a|\,(\frac{3a}{2} + a)}{(\frac{a}{2})^2 a^2} = \frac{|x-a|\,(\frac{5a}{2})}{\frac{a^2}{4} \cdot a^2} = \frac{|x-a|(\frac{5a}{2})}{\frac{a^4}{4}}$$

$$= \left(\frac{4}{a^4}\right)\left(\frac{5a}{2}\right)|x-a| = \frac{20a}{2a^4}|x-a| = \frac{10}{a^3}|x-a|$$

BIN: for $|x-a| < \frac{a}{2}$, $\left|\frac{1}{x^2} - \frac{1}{a^2}\right| \leq \frac{10}{a^3}|x-a|$

now solve $\frac{10}{a^3}|x-a| < \varepsilon$ for $|x-a|$

$$|x-a| < \frac{a^3}{10}\varepsilon$$

$$\boxed{\text{so } \delta(\varepsilon) = \min\left\{\frac{a^3}{10}\varepsilon, \frac{a}{2}\right\}}$$

prove $|x-a| < \delta(\varepsilon) \Rightarrow |f(x) - f(a)| < \varepsilon$

$|x-a| < \frac{a^3}{10}\varepsilon$ and $|x-a| < \frac{a}{2}$

I can use BIN

$\frac{10}{a^3}|x-a| < \varepsilon$

By BIN, $\left|\frac{1}{x^2} - \frac{1}{a^2}\right| \leq \frac{10}{a^3}|x-a|$ so $\boxed{\left|\frac{1}{x^2} - \frac{1}{a^2}\right| < \varepsilon}$