MATH 226 Examination 1 April 28, 2009

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Give all details of your reasoning. Each problem is worth 25 points for the total of 100 points.

18

Problem 1. (a) Write without the absolute values the exact value of the expression

$$\left|\pi^e - e^\pi\right| . = e^\pi - \pi^e$$

(b) Write the following English sentence as an inequality involving absolute value:

$$\boxed{\text{The distance between a number } x \text{ and the number } -\frac{2}{3} \text{ is less than } \frac{1}{4}.}$$

Illustrate with a diagram on the number line.

Problem 2. (a) State the definition of the absolute value function.

(b) State all the properties of absolute value that you will need in (c). (No proofs are required, just the statements. You can not list any version of the triangle inequality here.)

(c) Prove that $|a+b| \leq |a| + |b|$ for all $a, b \in \mathbb{R}$.

Problem 3. (a) State the definition of

$$\lim_{x\to+\infty} f(x) = L .$$

(b) Use the definition of limit to prove that

$$\lim_{x\to+\infty} \frac{x}{x+\cos x} = ? .$$

Problem 4. (a) State the ϵ-δ definition of continuity of a function f at a point a.

(b) Use ϵ-δ definition of continuity to prove that the function

$$f(x) = \frac{1}{x^2}$$

is continuous on $(0, +\infty)$.

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Math 226 Exam 1

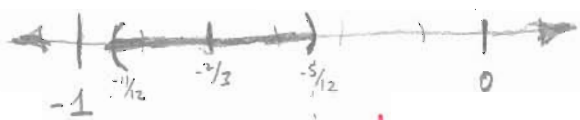
Problem 1:

(a) $|\pi^e - e^\pi| = \max\{\pi^e - e^\pi, -(\pi^e - e^\pi)\} = \max\{\pi^e - e^\pi, e^\pi - \pi^e\}$

$= e^\pi - \pi^e$

(b) $|x - (-\frac{2}{3})| < \frac{1}{4}$, $|x + \frac{2}{3}| < \frac{1}{4}$

Real number line:



x is within this open interval

$-\frac{2}{3} - \frac{1}{4} = \frac{-8-3}{12} = \frac{-11}{12}$

$-\frac{2}{3} + \frac{1}{4} = \frac{-8+3}{12} = \frac{-5}{12}$

Problem 2:

(a) $|x| := \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$

$\max\{u, -u\} = |u|$

(b) $x \leq |x|$, $-x \leq |x|$

(c) From the properties in (b), we see, for $a, b \in \mathbb{R}$, that $a \leq |a|$ & $b \leq |b|$, so $a + b \leq |a| + |b|$.

Also, $-a \leq |a|$ & $-b \leq |b|$, so $-a + (-b) \leq |a| + |b|$, $-(a+b) \leq |a| + |b|$.

Since both $a + b \leq |a| + |b|$ & $-(a+b) \leq |a| + |b|$, we can infer

$\max\{a+b, -(a+b)\} \leq |a| + |b|$. But, $\max\{a+b, -(a+b)\} = |a+b|$.

Therefore, $|a+b| \leq |a| + |b|$.

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Problem 3

(a) A real-valued function $f(x)$ has a limit $L \in \mathbb{R}$ if and only if the following two conditions are satisfied:

(I) $\exists X_0 \in \mathbb{R}$ such that $f(x)$ is defined for all $x \geq X_0$

(II) $\forall \varepsilon > 0$, $\exists X(\varepsilon)$ such that $X(\varepsilon) \in \mathbb{R}$, $X(\varepsilon) \geq X_0$, and

$$x > X(\varepsilon) \Rightarrow |f(x) - L| < \varepsilon.$$

(b) Prove $\lim_{x \to +\infty} \frac{x}{x + \cos x} = 1$.

If the two conditions above in part (a) are fulfilled, then the function has the limit:

(I) $f(x) = \frac{x}{x + \cos x}$ is undefined only when the denominator is zero. This does not occur when $x \geq 1$. So, let $X_0 = 1$. Then $f(x)$ is defined for all $x \geq X_0$

(II) we want to solve the inequality $|f(x) - L| < \varepsilon = \left|\frac{x}{x + \cos x} - 1\right| < \varepsilon$ for some relationship between x & ε:

$$\left|\frac{x}{x + \cos x} - 1\right| = \left|\frac{x - x - \cos x}{x + \cos x}\right| = \frac{|-\cos x|}{|x + \cos x|}$$

In "our world," for $x \geq 1$, the denominator is always positive:

$$\frac{|-\cos x|}{|x + \cos x|} = \frac{|-\cos x|}{x + \cos x}$$

Problem 3 continued...

(b) continued...

$|-\cos x| = |\cos x|$, by the properties of absolute value, & $|\cos x| \le 1$. So, $\dfrac{|\cos x|}{x+\cos x} \le \dfrac{1}{x+\cos x}$.

Also, for $x \ge 1$, $x + \cos x \ge x - 1$, so $\dfrac{1}{x+\cos x} \le \dfrac{1}{x-1}$.

Summarizing, for $x \ge 1$ $\dfrac{|\cos x|}{x+\cos x} = \left|\dfrac{x}{x+\cos x} - 1\right| \le \dfrac{1}{x-1}$

If $\frac{1}{x-1} < \varepsilon$, $x-1 > \frac{1}{\varepsilon}$, $x > \frac{1}{\varepsilon} + 1$.

Therefore, let $X(\varepsilon) = \max\{1, \frac{1}{\varepsilon} + 1\}$. Then $X(\varepsilon) \ge X_0$. because $X(\varepsilon) \ge 1$.

Assume that $x > X(\varepsilon)$. Then $x > \max\{1, \frac{1}{\varepsilon} + 1\}$, and so $x > 1$ & $x > \frac{1}{\varepsilon} + 1$. If $x > \frac{1}{\varepsilon} + 1$, $x - 1 > \frac{1}{\varepsilon}$, $\frac{1}{x-1} < \varepsilon$. Moreover, for $x > 1$ we demonstrated that

$\left|\dfrac{x}{x+\cos x} - 1\right| \le \dfrac{1}{x-1}$. It follows then that $\left|\dfrac{x}{x+\cos x} - 1\right| < \varepsilon$.

Therefore, $\lim_{x \to +\infty} \dfrac{x}{x+\cos x} = 1$.

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Problem 4

(a) A real valued function $f(x)$ is continuous at some $a \in \mathbb{R}$ if the following two conditions are satisfied:

(I) $\exists\ \delta_0 > 0$ such that $f(x)$ is defined for all $x \in (a - \delta_0, a + \delta_0)$.

(II) $\forall\ \varepsilon > 0$, $\exists\ \delta(\varepsilon) \in \mathbb{R}$ such that $0 < \delta(\varepsilon) \le \delta_0$ and

$$|x - a| < \delta(\varepsilon) \Rightarrow |f(x) - f(a)| < \varepsilon$$

(b) $f(x) = \frac{1}{x^2}$ is continuous on $(0, +\infty)$ if $f(x)$ is continuous at every point $\in (0, +\infty)$.

So, let $a \in (0, +\infty)$ be arbitrary. We will now show that the above two conditions are fulfilled, and so f is continuous at a:

(I) $f(x)$ is defined for $\mathbb{R}/\{0\}$. So, as long as zero is not in our interval $(a - \delta_0, a + \delta_0)$, we are safe. Let $\delta_0 = \frac{a}{2}$. Since $a > 0$, $\frac{a}{2} > 0$ & so $\delta_0 > 0$. Also, f is defined for all $x \in \left(a - \frac{a}{2}, a + \frac{a}{2}\right) = \left(\frac{a}{2}, \frac{3a}{2}\right)$.

(II) we want to solve $\left|\frac{1}{x^2} - \frac{1}{a^2}\right| < \varepsilon$ for $|x - a|$:

$$\left|\frac{1}{x^2} - \frac{1}{a^2}\right| = \left|\frac{a^2 - x^2}{x^2 a^2}\right| = \left|\frac{(a+x)(a-x)}{x^2a^2}\right| = \frac{|(a+x)(a-x)|}{|x^2a^2|}$$

$$= \frac{|a+x||a-x|}{|x^2a^2|}$$

In our interval of concern, $x \in \left(\frac{a}{2}, \frac{3a}{2}\right)$, $x > 0$ because $a > 0$. So, $|x^2a^2| = x^2a^2$ & $|a+x| = a+x$

$$\frac{|a+x||a-x|}{|x^2a^2|} = \frac{(a+x)|a-x|}{x^2a^2}$$

Problem 4 continued...

(b) continued...

By the properties of absolute value, $|a-x| = |-(a-x)| = |x-a|$

So we have $\dfrac{(a+x)|x-a|}{x^2a^2}$. On our interval, $x^2a^2 \geq \left(\frac{a}{2}\right)^2 a^2$,

$x^2a^2 \geq \frac{a^4}{4}$, so $\dfrac{1}{x^2a^2} \leq \dfrac{4}{a^4}$, and since $(a+x)|x-a| > 0$,

$\dfrac{(a+x)|x-a|}{x^2a^2} \leq \dfrac{4(a+x)|x-a|}{a^4}$. Also, for our interval, $a+x \leq a+\frac{3a}{2}$,

so $\dfrac{4(a+x)|x-a|}{a^4} \leq \dfrac{4\left(a+\frac{3a}{2}\right)|x-a|}{a^4} = \dfrac{10|x-a|}{a^3}$

If $\dfrac{10|x-a|}{a^3} < \varepsilon$, $|x-a| < \dfrac{\varepsilon a^3}{10}$.

So, let $\delta(\varepsilon) = \min\left\{\frac{a}{2}, \frac{\varepsilon a^3}{10}\right\}$. Then $\delta(\varepsilon) \leq \frac{a}{2}$, so $\delta(\varepsilon) \leq \delta_0$.

Assume $|x-a| < \delta(\varepsilon)$. Then $|x-a| < \frac{a}{2}$ and $-\frac{a}{2} < x-a < \frac{a}{2}$,

$\frac{a}{2} < x < \frac{3a}{2}$, which is to say $x \in \left(\frac{a}{2}, \frac{3a}{2}\right)$.

Also, $|x-a| < \dfrac{\varepsilon a^3}{10}$. Then $\dfrac{10|x-a|}{a^3} < \varepsilon$. But we showed

that for $x \in \left(\frac{a}{2}, \frac{3a}{2}\right)$, $\left|\frac{1}{x^2} - \frac{1}{a^2}\right| = \dfrac{(a+x)|x-a|}{x^2a^2} \leq \dfrac{10|x-a|}{a^3}$

It follows that $\left|\frac{1}{x^2} - \frac{1}{a^2}\right| < \varepsilon$.

Therefore, $f(x)$ is continuous at a. And, since $a \in (0, +\infty)$ was arbitrary, it follows that $f(x)$ is continuous for all $x \in (0, +\infty)$.