MATH 226 Examination 2 May 29, 2009

Name Key.

Give all details of your reasoning. Each problem is worth 25 points for the total of 100 points.

Problem 1. (a) State the definition of a convergent sequence.

(b) State the definition of a bounded sequence.

(c) Prove that a convergent sequence is bounded.

Problem 2. (a) Consider the sequence

$$x_1 = 1, \qquad x_{n+1} = 1 + \frac{2}{3}\,x_n, \quad n \in \mathbb{N}.$$

Prove that this sequence converges and find its limit.

(b) State clearly which important theorem you used in your proof.

Problem 3. Give a detailed explanation of the following equality

$$3.142857142857142857142857142857\ldots = \frac{\text{????integer}}{\text{????integer}}\,.$$

In your explanation you <u>must</u> use the concept of geometric series. Simplify the fraction in the final answer.

Problem 4. Use convergence tests to decide whether the series is convergent or divergent. Explain your reasoning: State clearly which test is being used and make sure that all requirements of that test are fulfilled.

$$\text{(A)}\quad \sum_{n=1}^{+\infty} \frac{1}{\pi^n - e^n}\,; \qquad \text{(B)}\quad \sum_{n=1}^{+\infty} \frac{1}{n}\cos\Big(\frac{1}{n}\Big)\,; \qquad \text{(C)}\quad \sum_{n=1}^{+\infty} \Big(\sin\Big(\frac{1}{n}\Big)\Big)^2.$$

1 (a) A sequence $\{a_n\}_{n=1}^{\infty}$ converges if there exists $L \in \mathbb{R}$ such that for every $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{R}$ such that

$$n \in \mathbb{N} \text{ and } n > N(\varepsilon) \Rightarrow |a_n - L| < \varepsilon.$$

$\boxed{1}$

(b) A sequence $\{a_n\}_{n=1}^{\infty}$ is bounded if there exist $m, M \in \mathbb{R}$ such that

$$m \leq a_n \leq M \text{ for all } n \in \mathbb{N}.$$

(c) Assume that a sequence $\{a_n\}_{n=1}^{\infty}$ converges to L. Then there exists $N(1) \in \mathbb{R}$ (we take $\varepsilon = 1 > 0$ in the above definition) such that

$$|a_n - L| < 1 \text{ for all } n \in \mathbb{N}, n > N(1).$$

Hence $-1 + L < a_n < 1 + L$ for all $n \in \mathbb{N}$, $n \geq \lceil N(1) \rceil + 1$. Set $m = \min\{a_1, \dots, a_{\lceil N(1) \rceil}, -1 + L\}$ and $M = \max\{a_1, \dots, a_{\lceil N(1) \rceil}, L + 1\}$. Then clearly $m \leq a_n \leq M$ for all $n \in \mathbb{N}$.

2 (a) This sequence is bounded above by 3 and it is an increasing sequence. Proof is by Mathematical induction. $\boxed{2}$

We will prove: $\boxed{x_n < 3}$ $\forall n \in \mathbb{N}$.

The statement $\boxed{x_1 < 3}$ is true.

Now let $k \in \mathbb{N}$ and assume that $x_k < 3$.

Then $\frac{2}{3} x_k < 2$ and consequently

$x_{k+1} = 1 + \frac{2}{3} x_k < 3$. Thus $x_{k+1} < 3$ is proved. By Math. Ind. $x_n < 3$ is true for all $n \in \mathbb{N}$.

$x_n < 3$ implies $\frac{1}{3} x_n < 1$. Hence

$\frac{1}{3} x_n + \frac{2}{3} x_n < 1 + \frac{2}{3} x_n$. Thus $x_n < 1 + \frac{2}{3} x_n$.

That is $x_n < x_{n+1}$ for all $n \in \mathbb{N}$.

So the sequence is ~~mono~~ increasing.

By the statement in (b) this sequence converges: $\lim_{n \to \infty} x_n = L$. L must satisfy $L = 1 + \frac{2}{3} L$. Hence $L = 3$.

(b) The most important theorem used in (a) is: $\boxed{3}$

If a sequence is monotonic and bounded, then it converges.

This statement is a consequence of the completeness axiom.

3) $3.\overline{142857} = 3 + 0.\overline{142857}$

$$0.\overline{142857} = \frac{142857}{10^6} + \frac{142857}{(10^6)^2} + \cdots$$

$$= \sum_{j=1}^{\infty} \frac{142857}{(10^6)^j}$$

This is a geometric series with

$$a = \frac{142857}{10^6} \text{ and } r = \frac{1}{10^6}$$

Hence

$$0.\overline{142857} = \frac{142857}{10^6} \frac{1}{1 - \frac{1}{10^6}} =$$

$$= \frac{142857}{\cancel{10^6}} \frac{\cancel{10^6}}{999999} = \frac{142857}{999999} = \frac{1}{7}$$

Hence $3.\overline{142857} = \dfrac{22}{7}$.

④ (A) We use the ratio test. $\boxed{14}$

$\pi^n - e^n > 0$ for all $n \in \mathbb{N}$.

$$\frac{a_{n+1}}{a_n} = \frac{\frac{1}{\pi^{n+1}-e^{n+1}}}{\frac{1}{\pi^n - e^n}} = \frac{\pi^n - e^n}{\pi^{n+1}-e^{n+1}} = \frac{\pi^n\left(1-\left(\frac{e}{\pi}\right)^n\right)}{\pi^{n+1}\left(1-\left(\frac{e}{\pi}\right)^{n+1}\right)}$$

$$= \frac{1}{\pi}\left\{\frac{1-\left(\frac{e}{\pi}\right)^n}{1-\left(\frac{e}{\pi}\right)^{n+1}} \xrightarrow[(n\to\infty)]{} \frac{1}{\pi} < 1.\right.$$

By the ratio test this series converges.

(B) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}\cos\left(\frac{1}{n}\right)$

limit comparison test

$$\lim_{n\to\infty} \frac{\frac{1}{n}\,1}{\frac{1}{n_1}\cos\left(\frac{1}{n}\right)} = 1$$

Since $\sum \frac{1}{n}$ diverges $\sum \frac{1}{n}\cos\left(\frac{1}{n}\right)$ diverges.

Assume $\lim \frac{a_n}{b_n} = L$, ~~then~~

If $\sum b_n$ converges, then $\sum a_n$ converges

(C) This is easy: $\boxed{5}$

$\sin x \leq x$ for all $x > 0$

Hence $\sin \frac{1}{n} < \frac{1}{n}$ for all $n \in \mathbb{N}$

so $\left(\sin \frac{1}{n}\right)^2 < \frac{1}{n^2}$ for all $n \in \mathbb{N}$.

By comparison since $\sum \frac{1}{n^2}$ converges,

$\sum_{n=1}^{\infty} \left(\sin \frac{1}{n}\right)^2$ converges as well.

There is an easier argument for (B).

Clearly if $0 < x < y < \pi/2$

$\cos x > \cos y$.

Since $0 < \frac{1}{n} \leq 1 < \pi/2$ for all $n \in \mathbb{N}$

we have $\cos\left(\frac{1}{n}\right) \geq \cos 1 \ \forall n \in \mathbb{N}$

Hence $\frac{1}{n} \cos\left(\frac{1}{n}\right) \geq (\cos 1) \frac{1}{n}$

Since $\sum_{n=1}^{\infty} \frac{\cos 1}{n}$ is a constant multiple of the harmonic series it diverges. Hence $\sum_{n=1}^{\infty} \frac{1}{n}\cos\left(\frac{1}{n}\right)$ diverges.