1 New limits from old

1.1 Squeeze theorems

In this section and in Section 1.3 we establish general properties of limits which are based on the formal definition of limit. These properties are stated as theorems.

Establishing theorems of this kind involves a major step forward in sophistication. Up to this point we have been trying to show that limits exist directly from the definition. Now for the first time we are going to **assume** that some limit exists (I refer to this in class as a *green* limit.) and try to make use of this information to establish the existence of some other limit (I refer to this in class as a *red* limit.). Remember that to establish the existence of a limit, we had to come up with a procedure for finding $\delta(\epsilon)$ that will work for any $\epsilon > 0$ that is given. If we assume the existence of a limit, then we are assuming the existence of such a procedure, though we may not know explicitly what it is. I refer to this as a *green* $\delta(\epsilon)$. It is this procedure we will need to use in order to construct a new procedure for the limit whose existence we are trying to establish. I refer to this as a *red* $\delta(\epsilon)$.

We start by considering squeeze theorems that resemble the role of BIN in previous sections. The following theorem is the Sandwich Squeeze Theorem.

Theorem 1.1.1. *Let f, g and h be given functions and let a and L be real numbers. Suppose that the following three conditions are satisfied.*

(1) $\lim_{x\to a} f(x) = L$,

(2) $\lim_{x\to a} h(x) = L$,

(3) *There exists $\eta_0 > 0$ such that $f(x), g(x)$ and $h(x)$ are defined for all $x \in \big(a-\eta_0, a\big)\cup\big(a, a+\eta_0\big)$ and*

$$f(x) \le g(x) \le h(x) \quad \text{for all} \quad x \in \big(a-\eta_0, a\big)\cup\big(a, a+\eta_0\big).$$

Then

$$\lim_{x\to a} g(x) = L.$$

Proof. Here we have three functions and three definitions of limits, one for each function. Therefore we have to deal with three δ-s. We shall give them appropriate names that will distinguish them from each other. Let us name them δ_f, δ_g and δ_h.

In the theorem it is assumed that $\lim_{x\to a} f(x) = L$. This means that we are given the fact that for each $\epsilon > 0$ there exists $\delta_f(\epsilon) > 0$ (that is, we are given a function $\delta_f(\epsilon)$) such that

$$0 < |x-a| < \delta_f(\epsilon) \quad \Rightarrow \quad |f(x) - L| < \epsilon. \tag{1.1.1}$$

In class I refer to these as a green $\delta_f(\cdot)$ and a green implication.

Since the theorem assumes that $\lim_{x\to a} h(x) = L$, we are also given that for each $\epsilon > 0$ there exists $\delta_h(\epsilon) > 0$ such that

$$0 < |x-a| < \delta_h(\epsilon) \quad \Rightarrow \quad |h(x) - L| < \epsilon. \tag{1.1.2}$$

Again we refer to these as a green $\delta_h(\cdot)$ and a green implication.

We need to prove that $\lim_{x \to a} g(x) = L$. Therefore, following the definition of limit, we have to show that the following conditions are satisfied:

(I) There exists a real number $\delta_{0,g} > 0$ such that $g(x)$ is defined for each x in the set $\big(a - \delta_{0,g}, a\big) \cup \big(a, a + \delta_{0,g}\big)$.

(II) For each real number $\epsilon > 0$ there exists a real number $\delta_g(\epsilon)$ such that $0 < \delta_g(\epsilon) \leq \delta_{0,g}$ and such that

$$0 < |x - a| < \delta_g(\epsilon) \quad \Rightarrow \quad |g(x) - L| < \epsilon. \tag{1.1.3}$$

Since we have to produce $\delta_{0,g}, \delta_g(\epsilon)$ and we have to prove the last implication, all of these objects are red.

Notice that η_0 in the theorem is green.

The objective here is to use the green objects to produce the red objects. We shall do that next. We put:

(I) $\delta_{0,g} = \eta_0$. By the assumption of the theorem $g(x)$ is defined for each x in the set $\big(a - \eta_0, a\big) \cup \big(a, a + \eta_0\big)$.

(II) For each real number $\epsilon > 0$, put

$$\delta_g(\epsilon) = \min\big\{\delta_f(\epsilon), \delta_h(\epsilon), \eta_0\big\}.$$

This is a beautiful expression since the red object is expressed in terms of the green objects.

It remains to prove the red implication (1.1.3) using the green implications and the assumptions of the theorem.

To prove (1.1.3), $\boxed{\text{assume that } 0 < |x - a| < \delta_g(\epsilon).}$ Then, clearly, $0 < |x - a| < \eta_0$. This is telling me that $x \neq a$ and that x is no further than η_0 from a. Consequently, $x \in \big(a - \eta_0, a\big) \cup \big(a, a + \eta_0\big)$. Therefore, by the assumption of the theorem

$$f(x) \leq g(x) \leq h(x).$$

Subtracting L from each term in this inequality, I conclude that

$$f(x) - L \leq g(x) - L \leq h(x) - L.$$

Using the property of the absolute value that $-|u| \leq u \leq |u|$ for each real number u, we conclude that

$$-|f(x) - L| \leq f(x) - L \leq g(x) - L \leq h(x) - L \leq |h(x) - L|. \tag{1.1.4}$$

From the assumption $0 < |x - a| < \delta_g(\epsilon)$, we conclude that $0 < |x - a| < \delta_f(\epsilon)$. By the green implication (1.1.1), this implies that $|f(x) - L| < \epsilon$ and therefore

$$-\epsilon < -|f(x) - L|. \tag{1.1.5}$$

From the assumption $0 < |x - a| < \delta_g(\epsilon)$, we conclude that $0 < |x - a| < \delta_h(\epsilon)$. By the green implication (1.1.2), this implies that

$$|h(x) - L| < \epsilon. \tag{1.1.6}$$

Putting together the inequalities (1.1.4), (1.1.5) and (1.1.6), we conclude that

$$-\epsilon < g(x) - L < \epsilon. \tag{1.1.7}$$

The inequalities in (1.1.7) are equivalent to

$$|g(x) - L| < \epsilon.$$

This proves that $0 < |x-a| < \delta_g(\epsilon)$ implies $|g(x)-L| < \epsilon$ and this is exactly the red implication (1.1.3). This completes the proof. □

The following theorem is the Scissors Squeeze Theorem.

Theorem 1.1.2. *Let f, g and h be given functions and let $a \in \mathbb{R}$ and $L \in \mathbb{R}$. Assume that*

(1) $\lim_{x \to a} f(x) = L$,

(2) $\lim_{x \to a} h(x) = L$,

(3) *There exists $\eta_0 > 0$ such that $f(x), g(x)$ and $h(x)$ are defined for all $x \in \bigl(a-\eta_0, a\bigr) \cup \bigl(a, a+\eta_0\bigr)$ and*

$$f(x) \leq g(x) \leq h(x) \quad \textit{for all} \quad x \in \bigl(a - \eta_0, a\bigr),$$

and

$$h(x) \leq g(x) \leq f(x) \quad \textit{for all} \quad x \in \bigl(a, a + \eta_0\bigr).$$

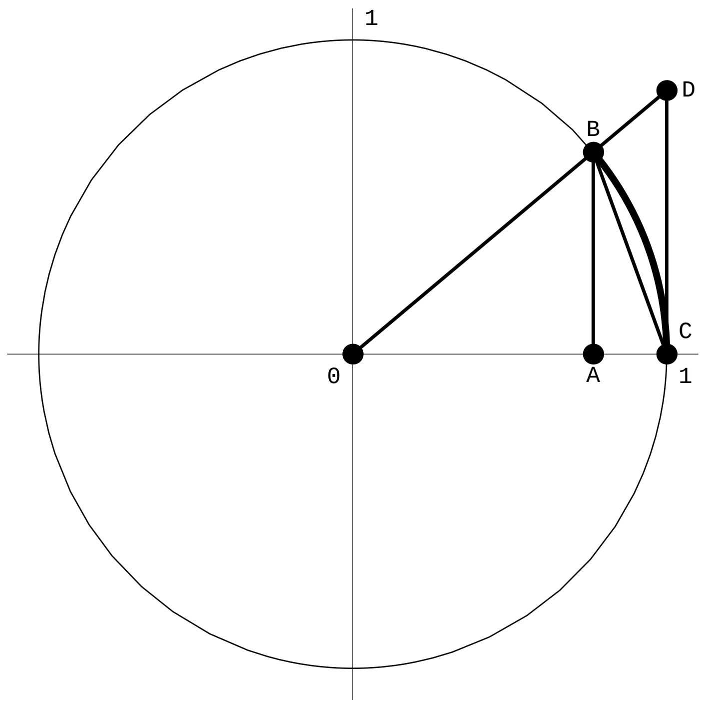
Then

$$\lim_{x \to a} g(x) = L.$$

1.2 Examples for squeeze theorems

The following picture and the numbers that *you can see on it* are essential for getting squeezes for limits involving trigonometric functions. The table to the left shows the numbers that you should be able to identify on the picture.

Geometric Object	Associated Number
Circular Arc from C to B	u
Line Segment $\overline{0A}$	$\cos u$
Line Segment $\overline{AB}$	$\sin u$
Line Segment $\overline{AC}$	$1 - \cos u$
Line Segment $\overline{CB}$	You Calculate
Line Segment $\overline{CD}$	$\tan u$
Line Segment $\overline{0B}$	1
Line Segment $\overline{0C}$	1



Example 1.2.1. Prove that $\lim_{x \to 0} \cos x = 1$.

Solution. Set $\eta_0 = \frac{\pi}{3}$. Consider positive u. Look at the picture above. The triangle $\triangle ACB$ is a right triangle. Therefore its hypothenuse, the line segment $\overline{CB}$, is longer than its side $\overline{AC}$ which equals to $1 - \cos u$. Thus

$$1 - \cos u = \overline{AC} \leq \overline{CB}. \tag{1.2.1}$$

The line segment $\overline{CB}$ is a segment of a straight line, therefore it is shorter than any other curve joining C and B. In particular it is shorter than the circular arc joining the points C and B. The length of this circular arc is u. Thus

$$\overline{CB} \leq \text{ Length of the Circular Arc from } C \text{ to } B \ (= \ u \ ). \tag{1.2.2}$$

Putting together the inequalities (1.2.1) and (1.2.2), we conclude that

$$1 - \cos u \leq u \quad \text{ for all } \quad 0 < u < \frac{\pi}{3}. \tag{1.2.3}$$

Since the length $\overline{0A} = \cos u$ is smaller than 1, from (1.2.3) we conclude that

$$0 \leq 1 - \cos u \leq u \quad \text{ for all } \quad 0 < u < \frac{\pi}{3} \ ,$$

or, equivalently,

$$1 - u \leq \cos u \leq 1 \quad \text{ for all } \quad 0 < u < \frac{\pi}{3} \ ,$$

Now we substitute $u = |x|$ and use the fact that $\cos |x| = \cos x$ and (1.2) becomes

$$1 - |x| \leq \cos x \leq 1 \quad \text{ for all } \quad -\frac{\pi}{3} < x < \frac{\pi}{3}.$$

This is a sandwich squeeze for $\cos x$. It is easy to prove that $\lim_{x\to 0} 1 = 1$ and $\lim_{x\to 0}\bigl(1-|x|\bigr) = 1$. (Please prove this using the definition!) Now the Sandwich Squeeze Theorem implies that $\lim_{x\to 0}\cos x = 1$. ☐

Example 1.2.2. Prove that $\lim_{x\to 0}\frac{\sin x}{x} = 1$.

Solution. To get a sandwich squeeze for this problem consider the following three areas on the picture above.

Area 1 The triangle $\triangle 0CB$.

Area 2 The segment of the unit disc bounded by the line segments $\overline{0C}$ and $\overline{0B}$ and the circular arc segment joining points C and B.

Area 3 The triangle $\triangle 0CD$.

The picture tells clearly the inequality between these areas. Write that inequality. Calculate each area in terms of the numbers that appear in the table above. This will lead to the inequality, which when simplified gives

$$\cos u \leq \frac{\sin u}{u} \leq 1 \quad \text{for all} \quad 0 < x < \frac{\pi}{3}. \tag{1.2.4}$$

Using the same idea as in the previous example, the inequality (1.2.4) leads to

$$\cos x \leq \frac{\sin x}{x} \leq 1 \quad \text{for all} \quad x \in \left(-\frac{\pi}{3}, 0\right) \cup \left(0, \frac{\pi}{3}\right). \tag{1.2.5}$$

The inequality (1.2.5) is exactly what we need in the Sandwich Squeeze Theorem. Please fill in all the details of the rest of the proof. ☐

Example 1.2.3. Prove that $\lim_{x\to 0}\frac{1-\cos x}{x^2} = \frac{1}{2}$.

Solution. To establish squeeze inequlaities consider three lengths:

Length 1 The line segment $\overline{AB}$.

Length 2 The line segment $\overline{CB}$.

Length 3 The length of a circular arc joining the points C and B.

The picture tells clearly the inequalities between these three lengths. Write these inequalities. Calculate each length in terms of the numbers that appear in the table above. This will lead to the inequalities, which, when simplified, give

$$\frac{1}{2}\left(\frac{\sin u}{u}\right)^2 \leq \frac{1-\cos u}{u^2} \leq \frac{1}{2} \quad \text{for all} \quad 0 < u < \frac{\pi}{3}. \tag{1.2.6}$$

From the inequality (1.2.6) and one inequality established in a previous example you can get an "easy" sandwich squeeze. Please fill in all the details of the rest of the proof. ☐

Example 1.2.4. Prove that $\lim_{x\to 0}\frac{\ln(1+x)}{x} = 1$.

Solution. The idea is to use the definition of ln as an integral and work with areas to get squeeze inequalities. ☐

1.3 Algebra of limits

A nickname that I gave to a function which has a limit L when x approaches a is: f is *constantish* L near a. If we are dealing with constant functions $f(x) = L$ and $g(x) = K$, then clearly the sum $f + g$ of these two functions is a constant function equal to $L + K$. The same is true for the product fg which is the constant function equal to LK. Another question is whether we can talk about the reciprocal $1/f$. If $L \neq 0$, then the reciprocal of f is defined and it equals $1/L$. In this section we shall prove that all these properties hold for constantish functions.

Theorem 1.3.1. *Let f, g, and h, be functions with domain and range in $\mathbb{R}$. Let a, K and L be real numbers. Assume that*

(1) $\lim\limits_{x\to a} f(x) = K$,

(2) $\lim\limits_{x\to a} g(x) = L$.

Then the following statements hold.

(A) *If $h = f + g$, then $\lim\limits_{x\to a} h(x) = K + L$.*

(B) *If $h = fg$, then $\lim\limits_{x\to a} h(x) = KL$.*

(C) *If $L \neq 0$ and $h = \dfrac{1}{g}$, then $\lim\limits_{x\to a} h(x) = \dfrac{1}{L}$.*

(D) *If $L \neq 0$ and $h = \dfrac{f}{g}$, then $\lim\limits_{x\to a} h(x) = \dfrac{K}{L}$.*

Proof. The assumption $\lim\limits_{x\to a} f(x) = K$ implies that

green(I-f) There exists (green!) $\delta_{0,f} > 0$ such that $f(x)$ is defined for all x in $\bigl(a - \delta_{0,f}, a\bigr) \cup \bigl(a, a + \delta_{0,f}\bigr)$;

green(II-f) For each $\epsilon > 0$ there exists (green!) $\delta_f(\epsilon)$ such that $0 < \delta_f(\epsilon) \leq \delta_{0,f}$ and such that

$$0 < |x - a| < \delta_f(\epsilon) \quad \Rightarrow \quad |f(x) - K| < \epsilon. \tag{1.3.1}$$

The assumption $\lim\limits_{x\to a} g(x) = L$ implies that

green(I-g) There exists (green!) $\delta_{0,g} > 0$ such that $g(x)$ is defined for all x in $\bigl(a - \delta_{0,g}, a\bigr) \cup \bigl(a, a + \delta_{0,g}\bigr)$;

green(II-g) For each $\epsilon > 0$ there exists (green!) $\delta_g(\epsilon)$ such that $0 < \delta_g(\epsilon) \leq \delta_{0,g}$ and such that

$$0 < |x - a| < \delta_g(\epsilon) \quad \Rightarrow \quad |g(x) - L| < \epsilon. \tag{1.3.2}$$

Proof of the statement (A). Remember that $h(x) = f(x) + g(x)$ here. First we list what is red in this proof.

red(I-h) There exists (red!) $\delta_{0,h} > 0$ such that $h(x)$ is defined for all x in $\big(a - \delta_{0,h}, a\big) \cup \big(a, a + \delta_{0,h}\big)$;

red(II-h) For each $\epsilon > 0$ there exists (red!) $\delta_h(\epsilon)$ such that $0 < \delta_h(\epsilon) \leq \delta_{0,h}$ and such that

$$0 < |x - a| < \delta_h(\epsilon) \quad \Rightarrow \quad |h(x) - (K + L)| < \epsilon. \tag{1.3.3}$$

I will not elaborate here how I got the idea for $\delta_{0,h}$ and $\delta_h(\epsilon)$, I will just give formulas and convince you that my choice is a correct one. The idea for the formulas comes from the boxed paragraph on page 8. I invite you to enjoy the separation of colors in the following formulas.

Let $\epsilon > 0$ be given. Put

$$\delta_{0,h} := \min\left\{\delta_{0,f}, \delta_{0,g}\right\}$$
$$\delta_h(\epsilon) := \min\left\{\delta_f\left(\frac{\epsilon}{2}\right), \delta_g\left(\frac{\epsilon}{2}\right)\right\}$$

Now I have to prove that $h(x)$ is defined for each $x \in \big(a - \delta_{0,h}, a\big) \cup \big(a, a + \delta_{0,h}\big)$. Assume that $x \in \big(a - \delta_{0,h}, a\big) \cup \big(a, a + \delta_{0,h}\big)$. Then

$$0 < |x - a| < \delta_{0,h} \leq \min\left\{\delta_{0,f}, \delta_{0,g}\right\}. \tag{1.3.4}$$

It follows from (1.3.4) that

$$0 < |x - a| < \delta_{0,f},$$

and therefore $x \in \big(a - \delta_{0,f}, a\big) \cup \big(a, a + \delta_{0,f}\big)$. Thus $f(x)$ is defined. It also follows from (1.3.4) that

$$0 < |x - a| < \delta_{0,g},$$

and therefore $x \in \big(a - \delta_{0,g}, a\big) \cup \big(a, a + \delta_{0,g}\big)$. Thus $g(x)$ is defined. Therefore $h(x) = f(x) + g(x)$ is defined for each $x \in \big(a - \delta_{0,h}, a\big) \cup \big(a, a + \delta_{0,h}\big)$.

Now I will prove the red implication (1.3.3). Assume

$$0 < |x - a| < \delta_h(\epsilon) = \min\left\{\delta_f\left(\frac{\epsilon}{2}\right), \delta_g\left(\frac{\epsilon}{2}\right)\right\}. \tag{1.3.5}$$

Then

$$0 < |x - a| < \delta_f\left(\frac{\epsilon}{2}\right). \tag{1.3.6}$$

The inequality (1.3.6) and the implication (1.3.1) allow me to conclude that

$$|f(x) - K| < \frac{\epsilon}{2}. \tag{1.3.7}$$

It follows from (1.3.5) that

$$0 < |x - a| < \delta_g\left(\frac{\epsilon}{2}\right). \tag{1.3.8}$$

The inequality (1.3.8) and the implication (1.3.2) allow me to conclude that

$$|g(x) - L| < \frac{\epsilon}{2}. \tag{1.3.9}$$

Now I remember that the absolute value has the property that $|u+v| \leq |u|+|v|$. I will apply this to the expression

$$|h(x)-(K+L)| = |f(x)+g(x)-K-L| = |\underbrace{(f(x)-K)}_{u} + \underbrace{(g(x)-L)}_{v}|$$

to get

$$|h(x)-(K+L)| \leq |f(x)-K| + |g(x)-L|. \tag{1.3.10}$$

This inequality plays a role of a BIN in this abstract proof. It has an unfriendly object on the left and all friendly objects on the right.

The inequalities (1.3.7), (1.3.9) and (1.3.10) imply that

$$|h(x)-(K+L)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \tag{1.3.11}$$

Reviewing my reasoning above you should be convinced that based on the assumption (1.3.5) I proved the inequality (1.3.11). This is exactly the implication (1.3.3). This completes the proof of the statement (A).

Proof of the statement (B). Remember that $h(x) = f(x)g(x)$ here. We first list what is red in this proof.

red(I-h) There exists (red!) $\delta_{0,h} > 0$ such that $h(x)$ is defined for all x in $\big(a-\delta_{0,h}, a\big) \cup \big(a, a+\delta_{0,h}\big)$;

red(II-h) For each $\epsilon > 0$ there exists (red!) $\delta_h(\epsilon)$ such that $0 < \delta_h(\epsilon) \leq \delta_{0,h}$ and such that

$$0 < |x-a| < \delta_h(\epsilon) \quad \Rightarrow \quad |h(x) - KL| < \epsilon. \tag{1.3.12}$$

I will not elaborate how I got the idea for $\delta_{0,h}$ and $\delta_h(\epsilon)$, I will just give formulas and convince you that my choice is a correct one. The idea for the formulas comes from the boxed paragraph on page 10. Again, I invite you to enjoy the separation of colors in the following formulas.

Let $\epsilon > 0$ be given. Put

$$\delta_{0,h} := \min\left\{\delta_{0,f}, \delta_g(1)\right\}$$
$$\delta_h(\epsilon) := \min\left\{\delta_f\left(\frac{\epsilon}{2(|L|+1)}\right), \delta_g\left(\frac{\epsilon}{2(|K|+1)}\right)\right\}$$

Now I have to prove that $h(x)$ is defined for each $x \in \big(a-\delta_{0,h}, a\big) \cup \big(a, a+\delta_{0,h}\big)$. Assume that $x \in \big(a-\delta_{0,h}, a\big) \cup \big(a, a+\delta_{0,h}\big)$. Then

$$0 < |x-a| < \delta_{0,h} \leq \min\left\{\delta_{0,f}, \delta_g(1)\right\}. \tag{1.3.13}$$

It follows from (1.3.13) that

$$0 < |x-a| < \delta_{0,f}\ ,$$

and therefore $x \in \big(a-\delta_{0,f}, a\big) \cup \big(a, a+\delta_{0,f}\big)$. Thus $f(x)$ is defined. It also follows from (1.3.13) that

$$0 < |x-a| < \delta_g(1). \tag{1.3.14}$$

Since by the assumption (II-g) I know that $\delta_g(1) \leq \delta_{0,g}$, the inequality (1.3.14) implies that

$$0 < |x-a| < \delta_{0,g}.$$

Therefore $x \in \big(a-\delta_{0,g}, a\big) \cup \big(a, a+\delta_{0,g}\big)$. Thus $g(x)$ is defined. Therefore $h(x) = f(x)g(x)$ is defined for each $x \in \big(a-\delta_{0,h}, a\big) \cup \big(a, a+\delta_{0,h}\big)$.

At this point I will prove another consequence of the inequality (1.3.14). This inequality and the implication (1.3.2) allow me to conclude that

$$|g(x) - L| < 1.$$

Therefore

$$-1 < g(x) - L < 1\ ,$$

or, equivalently

$$-1 + L < g(x) < L + 1.$$

Multiplying the last inequality by -1, I conclude that

$$-1 - L < -g(x) < -L + 1.$$

From the last two inequalities I conclude that $\max\{g(x), -g(x)\} < \max\{L+1, -L+1\} = \max\{L, -L\} + 1$. Thus

$$|g(x)| < |L| + 1. \tag{1.3.15}$$

Now I will prove the red implication (1.3.12). Assume

$$0 < |x-a| < \delta_h(\epsilon) = \min\left\{ \delta_f\left(\frac{\epsilon}{2(|L|+1)}\right), \delta_g\left(\frac{\epsilon}{2(|K|+1)}\right) \right\}. \tag{1.3.16}$$

Then

$$0 < |x-a| < \delta_f\left(\frac{\epsilon}{2(|K|+1)}\right). \tag{1.3.17}$$

The inequality (1.3.17) and the implication (1.3.1) allow me to conclude that

$$|f(x) - K| < \frac{\epsilon}{2(|L|+1)}. \tag{1.3.18}$$

It follows from (1.3.16) that

$$0 < |x-a| < \delta_g\left(\frac{\epsilon}{2(|K|+1)}\right). \tag{1.3.19}$$

The inequality (1.3.19) and the implication (1.3.2) allow me to conclude that

$$|g(x) - L| < \frac{\epsilon}{2(|K|+1)}. \tag{1.3.20}$$

Now I remember that the absolute value has the property that $|u+v| \le |u| + |v|$ and that $|uv| = |u||v|$. I will apply these properties to the expression

$$\begin{aligned}|h(x) - KL| = |f(x)g(x) - KL| &= |\underbrace{\big(f(x)g(x) - Kg(x)\big)}_{u} + \underbrace{\big(Kg(x) - KL\big)}_{v}| \\ &\le |f(x)g(x) - Kg(x)\big)| + |Kg(x) - KL| \\ &\le |g(x)|\,|f(x) - K| + |K|\,|g(x) - L|.\end{aligned}$$

Summarizing

$$|h(x) - KL| \le |g(x)|\,|f(x) - K| + |K|\,|g(x) - L|. \tag{1.3.21}$$

The inequalities (1.3.15) and (1.3.21) imply that

$$|h(x) - KL| \le \big(|L| + 1\big)\,|f(x) - K| + |K|\,|g(x) - L|. \tag{1.3.22}$$

This inequality plays a role of a BIN in this abstract proof. It has an unfriendly object on the left and all friendly objects on the right.

The inequalities (1.3.18), (1.3.20) and (1.3.22) imply that

$$|h(x) - LK| \le \big(|L| + 1\big)\,\frac{\epsilon}{2(|L|+1)} + |K|\,\frac{\epsilon}{2(|K|+1)} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \tag{1.3.23}$$

I hope that my reasoning above convinces you that the assumption (1.3.16) implies the inequality (1.3.23). This is exactly the implication (1.3.12). This completes the proof of the part (B).

Proof of the statement (C). Here we assume that $L \neq 0$ and $h(x) = \dfrac{1}{g(x)}$. Next we list what is red in this proof.

red(I-h) There exists (red!) $\delta_{0,h} > 0$ such that $h(x)$ is defined for all x in $\big(a - \delta_{0,h}, a\big) \cup \big(a, a + \delta_{0,h}\big)$;

red(II-h) For each $\epsilon > 0$ there exists (red!) $\delta_h(\epsilon)$ such that $0 < \delta_h(\epsilon) \le \delta_{0,h}$ and such that

$$0 < |x - a| < \delta_h(\epsilon) \quad \Rightarrow \quad \left|\frac{1}{g(x)} - \frac{1}{L}\right| < \epsilon. \tag{1.3.24}$$

I will not elaborate how I got the idea for $\delta_{0,h}$ and $\delta_h(\epsilon)$, I will just give formulas and convince you that my choice is a correct one. The idea for the formulas comes from the boxed paragraph on page 12. Again, I invite you to enjoy the separation of colors in the following formulas.

Let $\epsilon > 0$ be given. Remember that it is assumed that $|L| > 0$. Put

$$\begin{aligned}\delta_{0,h} &:= \delta_g\left(\frac{|L|}{2}\right) \\ \delta_h(\epsilon) &:= \min\left\{\delta_g\left(\frac{\epsilon L^2}{2}\right), \delta_g\left(\frac{|L|}{2}\right)\right\}.\end{aligned}$$

Now I have to prove that $h(x)$ is defined for each $x \in \big(a - \delta_{0,h}, a\big) \cup \big(a, a + \delta_{0,h}\big)$. Assume that $x \in \big(a - \delta_{0,h}, a\big) \cup \big(a, a + \delta_{0,h}\big)$. Then

$$0 < |x - a| < \delta_{0,h} = \delta_g\left(\frac{|L|}{2}\right).$$

This inequality and the implication (1.3.2) allow me to conclude that

$$|g(x) - L| < \frac{|L|}{2}.$$

Therefore

$$-\frac{|L|}{2} < g(x) - L < \frac{|L|}{2}\ ,$$

or, equivalently

$$-\frac{|L|}{2} + L < g(x) < L + \frac{|L|}{2}.$$

Multiplying the last inequality by -1, I conclude that

$$-L - \frac{|L|}{2} < -g(x) < \frac{|L|}{2} - L.$$

From the last two displayed relationships I conclude that

$$\max\{g(x), -g(x)\} > \max\left\{L - \frac{|L|}{2}, -L - \frac{|L|}{2}\right\} = \max\{L, -L\} - \frac{|L|}{2}.$$

Thus

$$|g(x)| > |L| - \frac{|L|}{2} = \frac{|L|}{2} > 0. \tag{1.3.25}$$

Consequently, $g(x) \neq 0$. Therefore, $h(x) = \dfrac{1}{g(x)}$ is defined for all $x \in \big(a - \delta_{0,h}, a\big) \cup \big(a, a + \delta_{0,h}\big)$.

Now I will prove the red implication (1.3.24). Assume

$$0 < |x - a| < \delta_h(\epsilon) = \min\left\{\delta_g\left(\frac{\epsilon L^2}{2}\right), \delta_g\left(\frac{|L|}{2}\right)\right\}. \tag{1.3.26}$$

Then

$$0 < |x - a| < \delta_g\left(\frac{\epsilon L^2}{2}\right). \tag{1.3.27}$$

The inequality (1.3.27) and the implication (1.3.2) allow me to conclude that

$$|g(x) - L| < \frac{\epsilon L^2}{2}. \tag{1.3.28}$$

It also follows from (1.3.26) that

$$0 < |x - a| < \delta_g\left(\frac{|L|}{2}\right).$$

We already proved that this inequality implies (1.3.25). Therefore

$$\frac{1}{|g(x)|} < \frac{2}{|L|}. \tag{1.3.29}$$

This inequality is used at the last step in the sequence of inequalities below. In some sense this is an abstract version of a "pizza-party" play.

Using our standard tools, algebra, properties of the absolute value and the inequality (1.3.29) we get

$$\begin{aligned}\left|h(x) - \frac{1}{L}\right| &= \left|\frac{1}{g(x)} - \frac{1}{L}\right| = \left|\frac{L - g(x)}{g(x)L}\right| = \frac{|L - g(x)|}{|g(x)|\ |L|} \\ &= \frac{|g(x) - L|}{|g(x)|\ |L|} \leq \frac{1}{|g(x)|}\ \frac{|g(x) - L|}{|L|} \leq \frac{2}{|L|}\ \frac{|g(x) - L|}{|L|}.\end{aligned}$$

Summarizing

$$\left|\frac{1}{g(x)} - \frac{1}{L}\right| \leq \frac{2}{L^2}\ |g(x) - L|\,. \tag{1.3.30}$$

This inequality plays a role of a BIN in this abstract proof. It has an unfriendly object on the left and all friendly objects on the right.

The inequalities (1.3.28) and (1.3.30) imply that

$$\left|\frac{1}{g(x)} - \frac{1}{L}\right| \leq \frac{2}{L^2}\ \frac{\epsilon L^2}{2} = \epsilon. \tag{1.3.31}$$

I hope that my reasoning above convinces you that the assumption (1.3.26) implies the inequality (1.3.31). This is exactly the implication (1.3.24). This completes the proof of the part (C).

Proof of the statement (D). Here we assume that $L \neq 0$ and $h(x) = \dfrac{f(x)}{g(x)}$. We can prove the statement (D) by using the universal power of the statements (B) and (C). First define the functions $g_1(x) = \dfrac{1}{g(x)}$. Then, by the statement (C) we know

$$\lim_{x \to a} g_1(x) = \frac{1}{L}. \tag{1.3.32}$$

Clearly, $h(x) = f(x)g_1(x)$. Now we can apply the statement (B) to this function h. Taking into account (1.3.32) the statement (B) implies

$$\lim_{x \to a} h(x) = K\,\frac{1}{L} = \frac{K}{L}.$$

This completes the proof of the statement (D). The theorem is proved. □

Exercise 1.3.2. Use the algebra of limits to give much simpler proofs for most of the limits in the previous exercises and examples.

1.4 L'Hospital Rule

By definition $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$.

Theorem 1.4.1. *Let f and g be functions and let a be a real number such that $f(a) = g(a) = 0$. Assume that the derivatives $f'(a)$ and $g'(a)$ exist and $g'(a) \neq 0$. Then*

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

Proof. Assume that the limits $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ and $g'(a) = \lim_{x \to a} \frac{g(x) - g(a)}{x - a}$ exist and $g'(a) \neq 0$. Then the limit

$$\lim_{x \to a} \frac{\frac{f(x) - f(a)}{x - a}}{\frac{g(x) - g(a)}{x - a}} \tag{1.4.1}$$

exists and it equals $\frac{f'(a)}{g'(a)}$. Remember that $f(a) = g(a) = 0$ and simplify

$$\frac{\frac{f(x) - f(a)}{x - a}}{\frac{g(x) - g(a)}{x - a}} = \frac{\frac{f(x)}{x - a}}{\frac{g(x)}{x - a}} = \frac{f(x)}{g(x)}. \tag{1.4.2}$$

Based on (1.4.1) and (1.4.2) I conclude that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

□

The following is a more powerful version of the L'Hospital rule. It's proof is not that much more complicated, but we will skip it here.

Theorem 1.4.2. *Let f and g be functions and let a be a real number such that $f(a) = g(a) = 0$. Assume that there exists $\delta_0 > 0$ such that $f(x), g(x), f'(x), g'(x)$ are defined for all $x \in \big(a - \delta_0, a\big) \cup \big(a, a + \delta_0\big)$. Assume that*

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = L.$$

Then $\lim_{x \to a} \frac{f(x)}{g(x)} = L$.

Example 1.4.3. Calculate $\lim_{x \to 0} \frac{x - \sin x}{x^3}$.

Solution. Put $f(x) = x - \sin x$ and $g(x) = x^3$. Put $\delta_0 = 1$. Then $f(x)$ and $g(x)$ is defined for all $x \in (-1, 1)$. Let $x \in (-1, 1)$ and calculate $f'(x) = 1 - \cos x$ and $g'(x) = 3x^2$. Now calculate

$$\begin{aligned} \lim_{x \to 0} \frac{f'(x)}{g'(x)} &= \lim_{x \to 0} \frac{1 - \cos x}{3x^2} = \lim_{x \to 0} \frac{1}{3} \cdot \frac{1 - \cos x}{x^2} \\ &= \frac{1}{3} \cdot \lim_{x \to 0} \frac{1 - \cos x}{x^2} = \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6} \end{aligned}$$

□

Exercise 1.4.4. Use the L'Hospital Rule to find each of the following limits.

(a) $\displaystyle\lim_{x \to 1} \frac{x^9 - 1}{x^5 - 1}$ (b) $\displaystyle\lim_{x \to 1} \frac{x^a - 1}{x^b - 1}$ (c) $\displaystyle\lim_{x \to \pi/2} \frac{1 - \sin x}{\cos x}$

(d) $\displaystyle\lim_{x \to 1} \frac{\ln x}{x - 1}$ (e) $\displaystyle\lim_{x \to 0} \frac{1 - \cos x}{(\sin x)^2}$ (f) $\displaystyle\lim_{x \to 0} \frac{\ln(1 + x)}{\sin x}$

(g) $\displaystyle\lim_{x \to 0} \frac{e^x - 1}{x}$ (h) $\displaystyle\lim_{x \to 0} \frac{e^x - 1 - x}{x^2}$ (i) $\displaystyle\lim_{x \to 0} \frac{x + \tan x}{\sin x}$