6 Continuous functions

6.1 Definition and examples

All this work about limits will now pay off since we shall be able to give mathematically rigorous definition of a continuous function.

Definition 6.1.1. Let f be a real valued function of a real variable and let a be a real number. The function f is continuous at a if the following two conditions are satisfied:

(i) The function f is defined at a, that is $f(a)$ is defined.

(ii) $\lim\limits_{x \to a} f(x) = f(a)$.

To understand Definition 6.1.1 the reader has to understand the concept of limit. Sometimes it is useful to state the definition of continuity directly, without appealing to the concept of limit.

Definition 6.1.2. Let f be a real valued function of a real variable and let a be a real number. The function f is continuous at a if the following two conditions are satisfied:

(I) There exists a $\delta_0 > 0$ such that $f(x)$ is defined for all $x \in (a - \delta_0, a + \delta_0)$.

(II) For each $\epsilon > 0$ there exists $\delta(\epsilon)$ such that $0 < \delta(\epsilon) \leq \delta_0$ and such that

$$|x - a| < \delta(\epsilon) \quad \Rightarrow \quad |f(x) - f(a)| < \epsilon.$$

Definition 6.1.2 is called ϵ-δ definition of continuity.

Definition 6.1.3. Let I be an interval in $\mathbb{R}$. A function f is *continuous on* I if it is continuous at each point in I.

Example 6.1.4. Let c be a real number and define $f(x) = c$ for all $x \in \mathbb{R}$. Use Definition 6.1.2 to prove that f is continuous at an arbitrary real number a.

Example 6.1.5. Let $f(x) = x$ for all $x \in \mathbb{R}$. Use Definition 6.1.2 to prove that f is continuous at an arbitrary real number a.

Example 6.1.6. Use ϵ-δ definition of continuity, that is Definition 6.1.2, to prove that the function $f(x) = 1/x$ is continuous on the interval $(0, +\infty)$.

Solution. Let $a \in (0, +\infty)$, that is let a be an arbitrary positive number. Chose $\delta_0 = a/2$. Since $a > 0$, we conclude that $a/2 > 0$ and $f(x) = 1/x$ is defined for all $x \in \big(a/2, 3a/2\big)$.

Let $\epsilon > 0$ be arbitrary. Now we have to solve

$$\left| \frac{1}{x} - \frac{1}{a} \right| < \epsilon \quad \text{for} \quad |x - a|.$$

First simplify the expression, using the fact that $x > 0$ and $a > 0$ and rules for the absolute value:

$$\left| \frac{1}{x} - \frac{1}{a} \right| = \left| \frac{a - x}{x\,a} \right| = \frac{|a - x|}{|x|\,|a|} = \frac{|x - a|}{x\,a}.$$

To get a larger expression which will be easy to solve we replace x in the denominator by the smallest possible value for x. That value is $a - a/2 = a/2$. This gives me my BIN:

$$\left|\frac{1}{x} - \frac{1}{a}\right| = \frac{|x-a|}{x\,a} \leq \frac{|x-a|}{\frac{a}{2}\,a} = 2\,\frac{|x-a|}{a^2}.$$

Thus my BIN is $\left|\frac{1}{x} - \frac{1}{a}\right| \leq 2\,\frac{|x-a|}{a^2}$ valid for all $x \in \bigl(a/2, 3a/2\bigr)$.

Solving the inequality $2\,\frac{|x-a|}{a^2} < \epsilon$ for $|x-a|$ is easy. The solution is $|x-a| < a^2\,\epsilon/2$. Now we define

$$\delta(\epsilon) = \min\left\{\frac{a^2\,\epsilon}{2}, \frac{a}{2}\right\}.$$

To finish the proof, it remains to prove the implication

$$|x-a| < \delta(\epsilon) \quad \Rightarrow \quad \left|\frac{1}{x} - \frac{1}{a}\right| < \epsilon.$$

This should be easy, using the BIN. □

Example 6.1.7. Use ϵ-δ definition of continuity, that is Definition 6.1.2, to prove that the function $x \mapsto \sqrt{x}$ is continuous on the interval $(0, +\infty)$.

Solution. Let $a \in (0, +\infty)$. Chose $\delta_0 = \frac{a}{2}$. Since $a > 0$, as before we conclude that $\frac{a}{2} > 0$ and the function $x \mapsto \sqrt{x}$ is defined for all $x \in (a/2, 3a/2)$.

Let $\epsilon > 0$ be arbitrary. Now we have to solve

$$\bigl|\sqrt{x} - \sqrt{a}\bigr| < \epsilon \quad \text{for} \quad |x-a|.$$

First simplify algebraically the expression, using the fact that $x > 0$ and $a > 0$ and rules for the absolute value.

$$\begin{aligned}\bigl|\sqrt{x} - \sqrt{a}\bigr| &= \left|\bigl(\sqrt{x} - \sqrt{a}\bigr)\,\frac{1}{1}\right| = \left|\bigl(\sqrt{x} - \sqrt{a}\bigr)\,\frac{\sqrt{x}+\sqrt{a}}{\sqrt{x}+\sqrt{a}}\right| = \left|\frac{x-a}{\sqrt{x}+\sqrt{a}}\right| \\ &= \frac{|x-a|}{|\sqrt{x}+\sqrt{a}|} = \frac{|x-a|}{\sqrt{x}+\sqrt{a}} \leq \frac{|x-a|}{\sqrt{a}}\end{aligned}$$

Thus my BIN is: $\bigl|\sqrt{x} - \sqrt{a}\bigr| \leq \frac{|x-a|}{\sqrt{a}}$, valid for $x > 0$.

Solving $\frac{|x-a|}{\sqrt{a}} < \epsilon$ for $|x-a|$ is easy: The solution is $|x-a| < \sqrt{a}\,\epsilon$. Now we define

$$\delta(\epsilon) = \min\left\{\sqrt{a}\,\epsilon, \frac{a}{2}\right\}.$$

It remains to prove the implication $|x-a| < \min\left\{\sqrt{a}\,\epsilon, \frac{a}{2}\right\} \quad \Rightarrow \quad |\sqrt{x} - \sqrt{a}| < \epsilon$. This should be easy, using the BIN. □

Example 6.1.8. Let $f(x) = \dfrac{1}{x^2+1}$ for all $x \in \mathbb{R}$. Use ϵ-δ definition to prove that f is continuous at an arbitrary $a \in \mathbb{R}$.

Example 6.1.9. Let a, b, c be any real numbers. Let $f(x) = ax^2 + bx + c$ for all $x \in \mathbb{R}$. Let v be an arbitrary real number. Prove that f is continuous at v.

Example 6.1.10. Let $f(x) = \sin x$ for all $x \in \mathbb{R}$. Prove that f is continuous at an arbitrary real number a.

Example 6.1.11. Let $f(x) = \cos x$ for all $x \in \mathbb{R}$. Prove that f is continuous at an arbitrary real number a.

HINT FOR Exercises 6.1.10 and 6.1.11. Let $A = (x_1, y_1)$ and $B = (x_2, y_2)$ be two points in the xy-plane. Then the length of the line segment $\overline{AB}$ is given by

$$\overline{AB} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

Consequently

$$|x_1 - x_2| \le \overline{AB} \qquad \text{and} \qquad |y_1 - y_2| \le \overline{AB}.$$

Let u and v be real numbers and set $A = (\cos u, \sin u)$, $B = (\cos v, \sin v)$. The last displayed inequalities now imply

$$|\cos u - \cos v| \le \overline{AB} \qquad \text{and} \qquad |\sin u - \sin v| \le \overline{AB}.$$

Recall that the points A and B are on the unit circle. Any two points on the unit circle determine two arcs. Denote by $\widehat{\mathrm{AB}}$ the length of the shorter circular arc determined by A and B. Since the shortest path between two points is a straight line we have that $\overline{AB} < \widehat{\mathrm{AB}}$. How is the arc length $\widehat{\mathrm{AB}}$ related to the numbers u and v? First, if $|u - v| \le \pi$, then $\widehat{\mathrm{AB}} = |u - v|$. Second, if $|u - v| > \pi$, then $\widehat{\mathrm{AB}} \le \pi < |u - v|$. Hence in each case $\widehat{\mathrm{AB}} \le |u - v|$. Thus we have established inequalities

$$|\cos u - \cos v| \le \overline{AB} \le \widehat{\mathrm{AB}} \le |u - v|,$$
$$|\sin u - \sin v| \le \overline{AB} \le \widehat{\mathrm{AB}} \le |u - v|,$$

for arbitrary real numbers u and v. These inequalities can be used to solve Exercises 6.1.10 and 6.1.11. THE END OF THE HINT.

Example 6.1.12. Let $f(x) = \ln x$ for all $x \in (0, +\infty)$. Prove that f is continuous at an arbitrary real number a.

Solution. Use the definition of ln to derive the squeeze for ln:

$$1 - \frac{1}{x} \le \ln x \le x - 1, \qquad 0 < x < +\infty.$$

Use the above squeeze to prove that for arbitrary $a > 0$ we have $\lim\limits_{x \to a} \ln\left(\dfrac{x}{a}\right) = 0$. Now use the rule for logarithms $\ln uv = \ln u + \ln v$. ☐

Example 6.1.13. Let $f(x) = e^x$ for all $x \in \mathbb{R}$. Prove that f is continuous at an arbitrary real number a.

Solution. Use the fact that $x \mapsto e^x$ is the inverse of the logarithm function to derive the squeeze for it:

$$1 + x \leq e^x \leq \frac{1}{1-x}, \qquad -\infty < x < 1.$$

Get the rest of the proof as an exercise. □

6.2 General theorems about continuous functions

The following theorem follows from Theorem 5.3.1.

Theorem 6.2.1 (Algebra of Continuous Functions)**.** *Let f and g be functions and let a be a real number. Assume that f and g are continuous at the point a.*

(a) *If $h = f + g$, then h is continuous at a.*

(b) *If $h = fg$, then h is continuous at a.*

(c) *If $h = \dfrac{f}{g}$ and $g(a) \neq 0$, then h is continuous at a.*

Example 6.2.2. Let $f(x) = \tan x$ for all $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$. Prove that f is continuous at an arbitrary real number a such that $-\dfrac{\pi}{2} < a < \dfrac{\pi}{2}$.

Solution. Use the algebra of continuous functions. □

The following theorem states that a composition of continuous functions is continuous.

Theorem 6.2.3. *Let f and g be functions and let a be a real number. Assume that g is continuous at a and that f is continuous at $g(a)$. If $h = f \circ g$, then h is continuous at a.*

Proof. Assume that the function g is continuous at a. That is assume

(I-g) There exists a $\delta_{0,g} > 0$ such that $g(x)$ is defined for all $x \in (a - \delta_{0,g}, a + \delta_{0,g})$.

(II-g) For each $\epsilon > 0$ there exists $\delta_g(\epsilon)$ such that $0 < \delta_g(\epsilon) \leq \delta_{0,g}$ and such that

$$|x - a| < \delta_g(\epsilon) \quad \Rightarrow \quad |g(x) - g(a)| < \epsilon.$$

Also assume that the function f is continuous at $g(a)$. That is assume

(I-f) There exists a $\delta_{0,f} > 0$ such that $f(x)$ is defined for all $x \in \big(g(a) - \delta_{0,f}, g(a) + \delta_{0,f}\big)$.

(II-f) For each $\epsilon > 0$ there exists $\delta_g(\epsilon)$ such that $0 < \delta_f(\epsilon) \leq \delta_{0,f}$ and such that

$$|u - g(a)| < \delta_f(\epsilon) \quad \Rightarrow \quad |f(u) - f(g(a))| < \epsilon.$$

Let $h = f \circ g$, that is $h(x) = f(g(x))$. I have to prove that h has the following properties: (These items are red.)

(I-h) There exists a $\delta_{0,h} > 0$ such that $h(x)$ is defined for all $x \in \big(a - \delta_{0,h}, a + \delta_{0,h}\big)$.

(II-h) For each $\epsilon > 0$ there exists $\delta_h(\epsilon)$ such that $0 < \delta_h(\epsilon) \leq \delta_{0,h}$ and such that

$$|x - a| < \delta_h(\epsilon) \quad \Rightarrow \quad |h(x) - h(a)| < \epsilon.$$

Where is h guaranteed to be defined? I must make sure that x is such that $|g(x) - g(a)| < \delta_{0,f}$. We can achieve this by using (II-g)!

Put $\delta_{0,h} := \delta_g\big(\delta_{0,f}\big)$. Now assume that $|x - a| < \delta_{0,h}$. By (II-g) it follows that $|g(x) - g(a)| < \delta_{0,f}$. Therefore $g(x) \in \big(g(a) - \delta_{0,f}, g(a) + \delta_{0,f}\big)$. Hence, by (I-f), $f(g(x))$ is defined. Thus we proved that $f(g(x))$ is defined whenever $|x - a| < \delta_{0,h}$.

Let $\epsilon > 0$ be given. Put $\delta_h(\epsilon) := \delta_g\big(\delta_f(\epsilon)\big)$. Now we prove the red implication in (II-h).

Assume $|x - a| < \delta_h(\epsilon)$. Then $|x - a| < \delta_g\big(\delta_f(\epsilon)\big)$. By the green implication in (II-g), we conclude that

$$|x - a| < \delta_g\big(\delta_f(\epsilon)\big) \quad \Rightarrow \quad |g(x) - g(a)| < \delta_f(\epsilon).$$

Using the green implication in (II-f), we conclude that

$$|g(x) - g(a)| < \delta_f(\epsilon) \quad \Rightarrow \quad |f(g(x)) - f(g(a))| < \epsilon.$$

Thus we proved that the assumption $|x - a| < \delta_h(\epsilon)$ implies that $|h(x) - h(a)| = |f(g(x)) - f(g(a))| < \epsilon$. This completes the proof. $\square$