Decimal expansions

The integers $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$ are called digits, or more precisely, decimal digits. We set $\mathbb{D} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

Theorem 1. *Let $d : \mathbb{N} \to \mathbb{D}$ be a sequence of digits. Prove that the series*

$$\sum_{k=1}^{+\infty} \frac{d_k}{10^k}$$

converges to a number in the interval $[0, 1]$

Proof. To prove that the series $\sum_{k=1}^{+\infty} \frac{d_k}{10^k}$ converges, we need to prove that the sequence of partial sums

$$S_n = \sum_{k=1}^{n} \frac{d_k}{10^k} \quad \text{for all} \quad n \in \mathbb{N},$$

converges. First observe that for all $n \in \mathbb{N}$ we have

$$S_{n+1} - S_n = \frac{d_{n+1}}{10^{n+1}} \geq 0.$$

(The last inequality holds since $d_{n+1} \geq 0$ and $10^{n+1} > 0$.) Consequently $S_n \leq S_{n+1}$ for all $n \in \mathbb{N}$, that is the sequence $S_n, n \in \mathbb{N}$, is nondecreasing. Next we will prove that this sequence is bounded above. For all $k \in \mathbb{N}$ we have that

$$d_k \leq 9$$

and therefore, since $10^k > 0$, we have

$$\frac{d_k}{10^k} \leq \frac{9}{10^k}.$$

Consequently, for all $n \in \mathbb{N}$ we have

$$S_n = \sum_{k=1}^{n} \frac{d_k}{10^k} \leq \sum_{k=1}^{n} \frac{9}{10^k}.$$

When we discussed geometric series we proved that

$$\sum_{k=1}^{n} \frac{9}{10^k} = \frac{9}{10} \frac{1 - \frac{1}{10^n}}{1 - \frac{1}{10}}.$$

Since clearly

$$\frac{9}{10} \frac{1 - \frac{1}{10^n}}{1 - \frac{1}{10}} < \frac{9}{10} \frac{1}{1 - \frac{1}{10}} = 1.$$

The last three displayed relations prove that

$$S_n < 1 \quad \text{for all} \quad n \in \mathbb{N}.$$

Since we proved that the sequence $S_n, n \in \mathbb{N}$, is nondecreasing and bounded above, the Monotone convergence theorem implies that this sequence converges. This proves that the series $\sum_{k=1}^{\infty} \frac{d_k}{10^k}$ converges. Denote the sum of this series by s, that is

$$s = \lim_{n \to +\infty} S_n.$$

We clearly have $0 \leq S_n \leq 1$ for all $n \in \mathbb{N}$. By Theorem 7.2.8 in the notes these inequalities imply

$$0 \leq s \leq 1.$$

This proves the theorem. □

If $d : \mathbb{N} \to \mathbb{D}$ is a sequence of digits and

$$s = \sum_{k=1}^{+\infty} \frac{d_k}{10^k},$$

then the series $\sum_{k=1}^{+\infty} \frac{d_k}{10^k}$ is called a decimal expansion of the number s. This series is commonly written as

$$s = 0.d_1 d_2 d_3 d_4 d_5 \ldots$$

A famous decimal expansion is

$$0.14159265358979323846264338327950288841971 \ldots .$$

Here

$$d_1 = 1, d_2 = 4, d_3 = 1, d_4 = 5, d_5 = 9, d_6 = 2, d_7 = 6, d_8 = 5, d_9 = 3, d_{10} = 5, d_{11} = 8, d_{12} = 9, \ldots .$$

In the next theorem we prove that each real number in $[0, 1)$ has a decimal expansion.

Theorem 2. *Let x be an arbitrary number in $[0, 1)$ and set*

$$d_n = \lfloor 10^n x \rfloor - 10 \lfloor 10^{n-1} x \rfloor \quad \text{for all} \quad n \in \mathbb{N}.$$

Then $d_n \in \mathbb{D}$ for all $n \in \mathbb{N}$ and

$$x = \sum_{k=1}^{+\infty} \frac{d_k}{10^k}.$$

Proof. First prove that $d_n \in \mathbb{D}$ for all $n \in \mathbb{N}$. For arbitrary $n \in \mathbb{N}$ we clearly have $d_n \in \mathbb{Z}$. We also have the following two inequalities

$$d_n = \lfloor 10^n x \rfloor - 10 \lfloor 10^{n-1} x \rfloor \geq \lfloor 10^n x \rfloor - 10\, 10^{n-1} x > -1$$

and

$$d_n = \lfloor 10^n x \rfloor - 10 \lfloor 10^{n-1} x \rfloor \leq 10^n x - 10 \lfloor 10^{n-1} x \rfloor = 10 \bigl(10^{n-1} x - \lfloor 10^{n-1} x \rfloor \bigr) < 10.$$

Thus, d_n is an integer and $-1 < d_n < 10$. This proves that $d_n \in \mathbb{D}$.

Let $n \in \mathbb{N}$ be arbitrary. Calculate

$$\begin{aligned}
S_n &= \sum_{k=1}^{n} \frac{d_k}{10^k} \\
&= \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} + \cdots + \frac{d_{n-1}}{10^{n-1}} + \frac{d_n}{10^n} \\
&= \frac{\lfloor 10x \rfloor - 10\lfloor x \rfloor}{10} + \frac{\lfloor 10^2 x \rfloor - 10\lfloor 10x \rfloor}{10^2} + \frac{\lfloor 10^3 x \rfloor - 10\,\lfloor 10^2 x \rfloor}{10^3} \\
&\qquad\qquad + \cdots + \frac{\lfloor 10^{n-1} x \rfloor - 10\lfloor 10^{n-2} x \rfloor}{10^{n-1}} + \frac{\lfloor 10^n\, x \rfloor - 10\,\lfloor 10^{n-1}\, x \rfloor}{10^n} \\
&= \left(\frac{\lfloor 10x \rfloor}{10} - 0 \right) + \left(\frac{\lfloor 10^2 x \rfloor}{10^2} - \frac{\lfloor 10x \rfloor}{10} \right) + \left(\frac{\lfloor 10^3 x \rfloor}{10^3} - \frac{\lfloor 10^2 x \rfloor}{10^2} \right) \\
&\qquad\qquad + \cdots + \left(\frac{\lfloor 10^{n-1} x \rfloor}{10^{n-1}} - \frac{\lfloor 10^{n-2} x \rfloor}{10^{n-2}} \right) + \left(\frac{\lfloor 10^n x \rfloor}{10^n} - \frac{\lfloor 10^{n-1} x \rfloor}{10^{n-1}} \right) \\
&= \frac{\lfloor 10^n x \rfloor}{10^n}.
\end{aligned}$$

Since

$$\lfloor 10^n x \rfloor \leq 10^n x < \lfloor 10^n x \rfloor + 1,$$

we have

$$0 \leq 10^n x - \lfloor 10^n x \rfloor < 1,$$

and therefore, since $10^n > 0$,

$$0 \leq x - \frac{\lfloor 10^n x \rfloor}{10^n} < \frac{1}{10^n}.$$

Consequently, for all $n \in \mathbb{N}$ we have

$$0 \leq x - S_n < \frac{1}{10^n}.$$

This is like a BIN. Let $\epsilon > 0$ be arbitrary. Set $N(\epsilon) = -\log \epsilon$. Then $n > N(\epsilon)$ implies $n > \log(1/\epsilon)$, and thus $10^n > 1/\epsilon$. Consequently $1/(10^n) < \epsilon$. Since, by BIN, $0 \leq x - S_n < 1/(10^n)$ we deduce that $|x - S_n| < \epsilon$. This proves that

$$x = \lim_{n \to +\infty} S_n = \sum_{k=1}^{+\infty} \frac{d_k}{10^k}. \qquad \square$$

The reasoning presented here can be adopted to any numeral system with a base b, where b is a natural number greater than 1. For example, in the hexadecimal numeral system the digits listed in increasing order are

$$\mathbb{D}_{\text{hex}} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F\}.$$

In the hexadecimal numeral system the famous decimal expansion mentioned before becomes

$$0.243F6A8885A308D313198A2E03707344A4093822\ldots\ ,$$

which is certainly not that famous.

Exercise 3. Some of us are lucky enough to have the initials which are hexadecimal digits. So, we might be curious which number has hexadecimal expansion $0.BCBCBCBCBC....$ For the full entertainment value, represent this number as a fraction of two natural numbers in hexadecimal numeral system.

Exercise 4. You might think that the previous exercise is somewhat egotistical. If that is the case, or in any case, repeat the previous exercise for the number $0.ABCABCABCABCABC....$ But in this case, please do not forget to simplify the fraction.