For most functions f a proof of $\lim_{x \to +\infty} f(x) = L$ based on the definition in the notes should consist from the following steps.

(1) Find X_0 such that $f(x)$ is defined for all $x \geq X_0$. Justify your choice.

(2) Use algebra to simplify the expression $|f(x) - L|$ with the assumption that $x \geq X_0$. Try to eliminate the absolute value.

(3) Use the simplification from (2) to discover a BIN:

$$\boxed{|f(x) - L| \leq b(x) \quad \text{valid for} \quad x \geq X_0}\ .$$

The content of the box above is a BIN.

Here $b(x)$ should be a simple function with the following properties:

(a) $b(x) > 0$ for all $x \geq X_0$.

(b) $b(x)$ is tiny for huge x.

(c) $b(x) < \epsilon$ is easily solvable for x for each $\epsilon > 0$. The solution should be of the form

$$x \ > \ \boxed{\text{some expression involving } \epsilon}.$$

Warning: In the above inequality $\boxed{\text{some expression involving } \epsilon}$ must be huge when ϵ is tiny.

(4) Use the solution of $b(x) < \epsilon$, that is $\boxed{\text{some expression involving } \epsilon}$, and X_0 to define

$$X(\epsilon) = \max\left\{X_0, \boxed{\text{some expression involving } \epsilon}\right\}.$$

(5) Use the BIN above to **prove** the implication $x > X(\epsilon) \Rightarrow |f(x) - L| < \epsilon$.

Note: The structure of this **proof** is always the same.

(i) First assume that $x > X(\epsilon)$.

(ii) The definition of $X(\epsilon)$ yields that

$$X(\epsilon) \geq X_0 \quad \text{and} \quad X(\epsilon) \geq \boxed{\text{some expression involving } \epsilon}.$$

(iii) Based of (5i) and (5ii) we conclude that the following two inequalities are true:

$$x > X_0 \quad \text{and} \quad x > \boxed{\text{some expression involving } \epsilon}.$$

(iv) From (3) part (c) we know that

$$x > \boxed{\text{some expression involving } \epsilon}$$

implies $b(x) < \epsilon$. Therefore (5iii) yields that $b(x) < \epsilon$ is true.

(v) We also established that the BIN is true:

$$\boxed{|f(x) - L| \leq b(x) \quad \text{valid for} \quad x \geq X_0}$$

(vi) Together $|f(x) - L| \leq b(x)$ and $b(x) < \epsilon$ yield

$$|f(x) - L| < \epsilon.$$

This is exactly what we needed to prove.