ON TWO COMMON SEQUENCES

BRANKO ĆURGUS

The following two sequences are commonly used to define the number e as their limit:

$$P_n = \left(1 + \frac{1}{n}\right)^n, \qquad n \in \mathbb{N},$$

$$S_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} = \sum_{k=0}^{n} \frac{1}{k!}, \quad n \in \mathbb{N}.$$

Here $\mathbb{N}$ denotes the set of all positive integers.

In this note we give a direct and easy-to-remember proof that the sequences $\{P_n\}$ and $\{S_n\}$ converge to the same limit.

1. Preliminaries

We first recall the binomial theorem which states that for all real numbers x and y, and all positive integers m,

$$(x+y)^m = \sum_{k=0}^{m} \binom{m}{k} x^{m-k} y^k,$$

where $\binom{m}{k} = \frac{m!}{k!(m-k)!}$ are binomial coefficients.

We will also use the familiar formula

$$1 + 2 + \cdots + k - 1 = \frac{(k-1)k}{2},$$

which, as the story goes (see [1] for an impressive detailed account), Carl Friedrich Gauss discovered shortly after his seventh birthday.

Further, we will use the following three limit theorems.

Squeeze Theorem. *If $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences such that $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$ and both $\{a_n\}$ and $\{c_n\}$ converge to the same limit L, then $\{b_n\}$ converges to L.*

Sum of Limits Theorem. *If $\{a_n\}$ converges to K and $\{b_n\}$ converges to L, then the sequence $\{a_n + b_n\}$ converges to $K + L$.*

Monotone Convergence Theorem. *An increasing sequence which is bounded above converges.*

2. Results

Proposition 1. *The sequence $\{S_n\}$ is increasing and bounded above by 3.*

Proof. The sequence $\{S_n\}$ is increasing since

$$S_{n+1} - S_n = \frac{1}{(n+1)!} > 0 \quad \text{for all} \quad n \in \mathbb{N}.$$

Clearly $S_1 < 3$. Further, notice that $1/k! \leq 1/\big((k-1)k\big)$ for all integers k with $k \geq 2$. Therefore, for all integers n greater than 1 we have

$$\begin{aligned} S_n &= \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{(n-1)!} + \frac{1}{n!} \\ &\leq 1 + 1 + \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{(n-2)(n-1)} + \frac{1}{(n-1)n} \\ &= 2 + \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n-2} - \frac{1}{n-1}\right) + \left(\frac{1}{n-1} - \frac{1}{n}\right) \\ &= 3 - \frac{1}{n} \\ &< 3. \end{aligned}$$

This proves that 3 is an upper bound for $\{S_n\}$. □

Proposition 2. *For all $n \in \mathbb{N}$ we have*

$$S_n - \frac{3}{2n} \leq P_n \leq S_n. \tag{1}$$

Proof. A straightforward calculations confirm that (1) is true for $n = 1$ and $n = 2$. Now let n be an integer greater than 2. The following proof of (1) is a succession of five steps each suggesting the next one.
1. The **binomial theorem** with $x = 1$, $y = 1/n$ and $m = n$ yields an expanded expression for P_n:

$$P_n = \left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!}\,\frac{1}{n^k} = 1 + 1 + \sum_{k=2}^{n} \frac{1}{k!}\,\frac{n!}{(n-k)!\,n^k}. \tag{2}$$

2. For $k \in \{2, \ldots, n\}$, we **rewrite the coefficient** with $1/k!$ in (2) as the product of $k - 1$ factors:

$$\begin{aligned} \frac{n!}{(n-k)!\,n^k} &= \frac{n(n-1)\cdots(n-k+1)}{n^k} \\ &= \left(\frac{n}{n}\right)\left(\frac{n-1}{n}\right)\left(\frac{n-2}{n}\right)\cdots\left(\frac{n-k+1}{n}\right) \\ &= \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{k-1}{n}\right). \end{aligned} \tag{3}$$

3. Notice that 1 is an **upper bound** for (3) since all the factors in (3) are positive and less then 1.
4. Next we look for a **lower bound** for the product in (3). We proceed recursively. At each step, in some sense, we turn a product into a sum. For $k = 2$ the product in (3) has only one term and obviously

$$\left(1 - \frac{1}{n}\right) \geq 1 - \frac{1}{n}.$$

Multiplying both sides by $\left(1-\frac{2}{n}\right)$, then expanding the right-hand side and dropping a positive term, we get a lower bound for $k=3$:

$$\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) \geq \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) > 1-\frac{1+2}{n}.$$

Now multiply both sides by $\left(1-\frac{3}{n}\right)$ we similarly get a lower bound for $k=4$:

$$\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right) > \left(1-\frac{1+2}{n}\right)\left(1-\frac{3}{n}\right) = 1-\frac{1+2+3}{n}.$$

Repeating this process a total of $k-1$ times yields:

$$\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right) > 1-\frac{1+\cdots+(k-1)}{n} = 1-\frac{(k-1)k}{2n}.$$

We record the upper and lower bound for the product in (3) as follows: For all $k\in\{2,\ldots,n\}$ we have

$$1-\frac{(k-1)k}{2n} < \frac{n!}{n^k(n-k)!} < 1. \tag{4}$$

5. We apply the inequalities from (4) to the most right expression in (2) to **establish the inequalities** for P_n:

$$1+1+\sum_{k=2}^{n}\frac{1}{k!}\left(1-\frac{(k-1)k}{2n}\right) < P_n < 1+1+\sum_{k=2}^{n}\frac{1}{k!}\cdot 1 = S_n. \tag{5}$$

A simplification of the left-hand side of (5) leads to

$$\sum_{k=0}^{n}\frac{1}{k!}-\sum_{k=2}^{n}\frac{1}{k!}\frac{(k-1)k}{2n} = S_n-\frac{1}{2n}\sum_{k=2}^{n}\frac{1}{(k-2)!} = S_n-\frac{1}{2n}S_{n-2}.$$

Further, since $S_{n-2}<3$, we also have

$$S_n-\frac{1}{2n}S_{n-2} > S_n-\frac{3}{2n}.$$

Consequently, the left-hand side of (5) is greater than $S_n-3/(2n)$. This proves (1) for all $n>2$ and completes the proof of the proposition. □

Theorem 3. *The sequences $\{P_n\}$ and $\{S_n\}$ converge to the same limit.*

Proof. Since by Proposition 1 the sequence $\{S_n\}$ is increasing and bounded above, the Monotone Convergence Theorem implies that it converges. The sequence $\{-2/(3n)\}$ converges to 0, by the Sum of Limits Theorem, the sequence $\{S_n-2/(3n)\}$ converges to the limit of $\{S_n\}$. Now, by Proposition 2 and the Squeeze Theorem the sequence $\{P_n\}$ converges to the the limit of $\{S_n\}$. □

Theorem 3 justifies the following definition.

Definition 4. The number e is the common limit of the sequences $\{P_n\}$ and $\{S_n\}$.

References

[1] B. Hayes, Versions of the Gauss schoolroom anecdote. Available at `http://www.sigmaxi.org/amscionline/gauss-snippets.html`