```
In[ ]:= Cosh[x]
Out[ ]= Cosh[x]
```

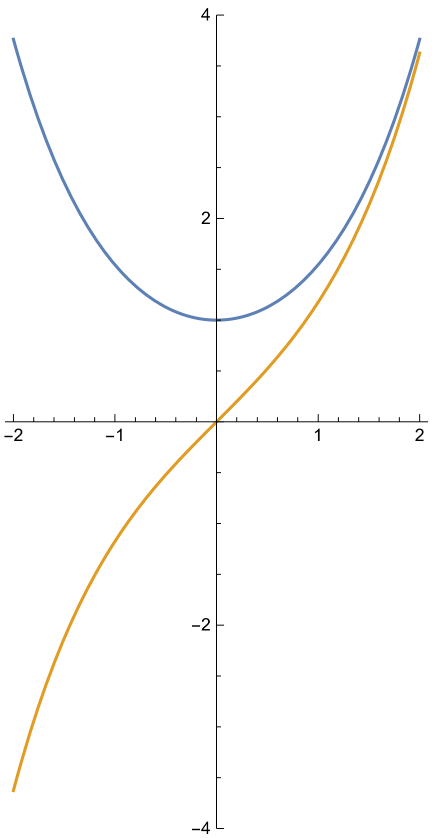
```
In[ ]:= Sinh[x]
Out[ ]= Sinh[x]
```

```
In[ ]:= Tanh[x]
Out[ ]= Tanh[x]
```

Below I show the hyperbolic cosine and hyperbolic sine function.

```
In[ ]:= Plot[{(Exp[x] + Exp[-x])/2, (Exp[x] - Exp[-x])/2},
         {x, -2, 2}, PlotRange → {-4, 4}, AspectRatio → Automatic]
```

Out[ ]=



The hyperbolic cosine Cosh[x] = $\frac{\text{Exp}[x]+\text{Exp}[-x]}{2}$ has the following properties

```
In[1]:= {Cosh[0], Cosh'[0]}
Out[1]= {1, 0}
```

which are identical to the analogous properties of Cos[x]

```
In[2]:= {Cos[0], Cos'[0]}
Out[2]= {1, 0}
```

In addition, the hyperbolic cosine is even as is cosine and the hyperbolic cosine satisfies the differential equation

In[3]:=
```
Cosh''[x] - Cosh[x] == 0
```
Out[3]=
```
True
```

While cosine satisfies the differential equation

In[4]:=
```
Cos''[x] + Cos[x] == 0
```
Out[4]=
```
True
```

Altogether Cosh and Cos share many properties, so they do deserve similar names. Moreover, if you take Math 438, Complex Analysis, you will learn even more similarities between these functions.

Similarly, the hyperbolic sine Sinh[*x*] = $\frac{\text{Exp}[x]-\text{Exp}[-x]}{2}$ has the following properties

In[5]:=
```
{Sinh[0], Sinh'[0]}
```
Out[5]=
```
{0, 1}
```

which are identical to the analogous properties of Sin[x]

In[6]:=
```
{Sin[0], Sin'[0]}
```
Out[6]=
```
{0, 1}
```

In addition, the hyperbolic sine is odd as is sine and the hyperbolic sine satisfies the differential equation

In[7]:=
```
Sinh''[x] - Sinh[x] == 0
```
Out[7]=
```
True
```

While sine satisfies the differential equation

In[8]:=
```
Sin''[x] + Sin[x] == 0
```
Out[8]=
```
True
```

Altogether Sinh and Sin share many properties, so they do deserve similar names. Moreover, if you take Math 438, Complex Analysis, you will learn even more similarities between these functions.

The next natural definition of a hyperbolic function is the hyperbolic tangent Tanh[x] which is defined as the fraction Sinh[x]/Cosh[x] :

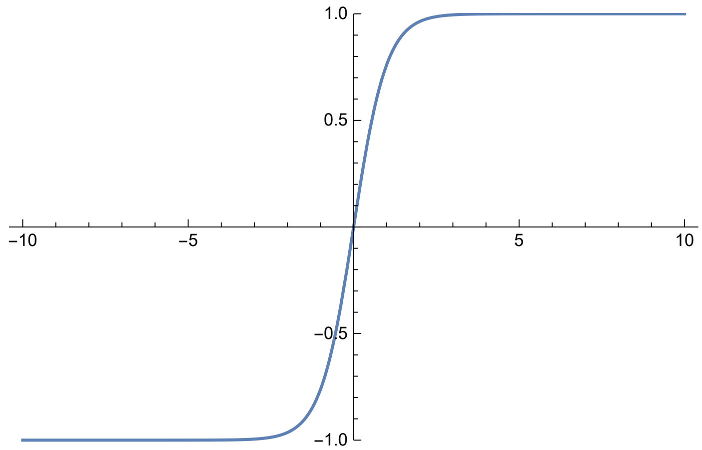
In[ ]:=
$$\frac{\frac{\text{Exp}[x]-\text{Exp}[-x]}{2}}{\frac{\text{Exp}[x]+\text{Exp}[-x]}{2}}$$

In[ ]:=
$$\frac{\frac{\text{Exp}[x]-\text{Exp}[-x]}{2}}{\frac{\text{Exp}[x]+\text{Exp}[-x]}{2}}$$

Out[ ]=
$$\frac{-e^{-x}+e^{x}}{e^{-x}+e^{x}}$$

In[∘]:=
```
Plot[{Tanh[x]}, {x, -10, 10}, PlotRange → {-1, 1}]
```

Out[∘]=



Tanh[x] is a famous function since it approaches 1 very quickly as x becomes large positive number.

In[∘]:=
```
Tanh[8]
```

Out[∘]= Tanh[8]

In[9]:=
```
N[Tanh[8], 10]
```

Out[9]= 0.9999997749

In[10]:=
```
N[Tanh[20], 10]
```

Out[10]= 1.000000000

Although, have in mind that Tanh[x] < 1 for all $x \in \mathbb{R}$.

In[∘]:=
```
Limit[Tanh[x], x → Infinity]
```

Out[∘]= 1

What engineers mean by saying Tanh[8]=1?

They mean that 1-error < Tanh[8] < 1+error

1-error < Tanh[8] < 1+error

equivalent to

1-error +(-1) < Tanh[8] -1 < 1+error +(-1)

equivalent to

-error < Tanh[8] -1 < +error error is always a positive number

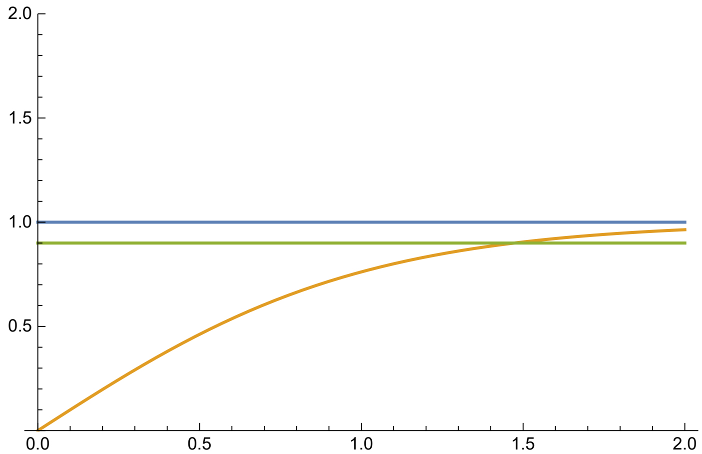
equivalent

| Tanh[8] - 1 | < error

In Calculus the error is always denoted by ε

In[•]:= `Plot[{1, Tanh[x], 1 - 0.1}, {x, 0, 2}, PlotRange → {0, 2}]`

Out[•]=



In[•]:= `N[Tanh[20], 20]`

Out[•]= 0.99999999999999999150

In[•]:= `N[Tanh[100], 200]`

Out[•]= 0.99999999999999999999999999999999999999999999999999999999999999999999999999999999999999\
723220694652652493870263708604183062919390483532104558121214929377512793909801402438240\
5865765139346117090256137123

For every $\varepsilon > 0$ there exists $X(\varepsilon)$ such that $x \geq X(\varepsilon)$ then we must have $| f(x) - L | < \varepsilon$

For every $\varepsilon > 0$ there exists $X(\varepsilon)$ such that $x \geq X(\varepsilon)$ implies $| f(x) - L | < \varepsilon$

If the statement in the preceding cell is true, then we say that L is the limit of $f(x)$ as x approaches $+\infty$.
lim_{$x \to +\infty$} $f(x) = L$

L has the following property: $\forall\ \varepsilon > 0\ \ \exists\ X(\varepsilon)$ s.t. $x \geq X(\varepsilon) \Rightarrow | f(x) - L | < \varepsilon$

In[•]:= `Limit[Tanh[x], x → Infinity]`

Out[•]= 1

To do a formal proof of a limit we need to start from the end. Let us do that for Tanh[x]

Let $\varepsilon > 0$ be arbitrary. I need to understand when is the following inequality true $| \tanh(x) - 1 | < \varepsilon$

First I simplify the expression

$$| \tanh(x) - 1 | = \left| \frac{\frac{e^x - e^{-x}}{2}}{\frac{e^x + e^{-x}}{2}} - 1 \right| = \left| \frac{e^x - e^{-x}}{e^x + e^{-x}} - 1 \right| = \left| \frac{e^x - e^{-x}}{e^x + e^{-x}} - \frac{e^x + e^{-x}}{e^x + e^{-x}} \right| = \left| \frac{e^x - e^{-x} - (e^x + e^{-x})}{e^x + e^{-x}} \right| = \left| \frac{-2\, e^{-x}}{e^x + e^{-x}} \right| = \frac{2\, e^{-x}}{e^x + e^{-x}}$$