One sided limit and squeeze theorems

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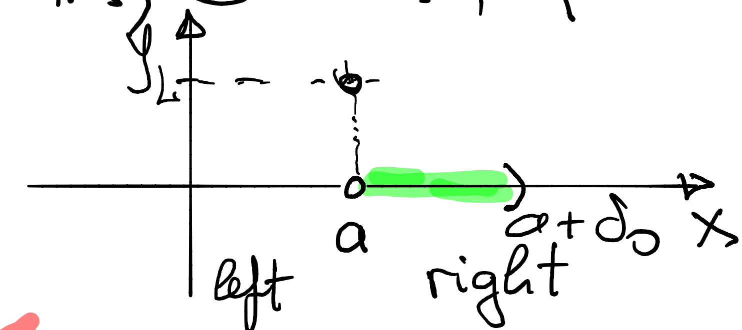
Def. $\boxed{\lim_{x \downarrow a} f(x) = L}$ finite limit as x approaches a from the right

$a \in \mathbb{R}$, $L \in \mathbb{R}$, $D \subseteq \mathbb{R}$, $f : D \to \mathbb{R}$



the function must be defined in the right neighbourhood of a

(I) $\exists\, \delta_0 > 0$ s.t. $(a, a+\delta_0) \subseteq D$

(II) $\forall \varepsilon > 0 \quad \exists\, \delta(\varepsilon) \leq \delta_0$ s.t.

$$0 < x - a < \delta(\varepsilon) \Rightarrow |f(x) - L| < \varepsilon$$

New limits from old

Squeeze Theorems

Sandwich Squeeze

In all of these theorems abstract functions appear: two kinds of abstract functions

friends and foes

we know some facts about these functions ⟷ we don't know much about these functions

Sandwich Squeeze: three functions: f, g, h

friends: f and h

foe: g

I know

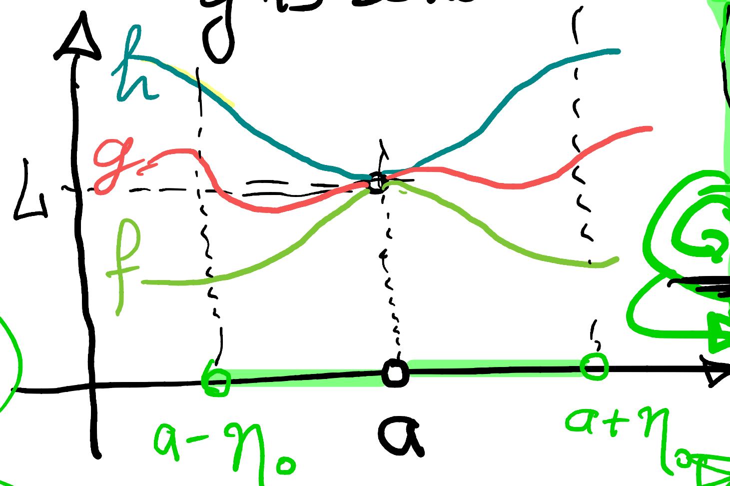
G1 $\lim\limits_{x\to a} f(x) = L$

G2 $\lim\limits_{x\to a} h(x) = L$

assumptions are here

the connection

" g is sandwitched between f & g "



$\exists\, \eta_0 > 0$

s.t.

$\forall x \in (a-\eta_0, a) \cup (a, a+\eta_0)$

G3 $\underbrace{f(x) \le g(x) \le h(x)}_{\text{squeeze}}$

Claim: $\boxed{\lim_{x \to a} g(x) = L}$

Proof. Assume G1, G2, G3

(I)$_f$ $\exists \delta_{0,f} > 0$ $(a-\delta_{0,f}, a) \cup (a, a+\delta_{0,f}) \subseteq D_f$

(II)$_f$ $\forall \varepsilon > 0$ $\exists \delta_f(\varepsilon) > 0$ s.t. $\delta_f(\varepsilon) \le \delta_{0,f}$

$0 < |x-a| < \delta_f(\varepsilon) \Rightarrow |f(x) - L| < \varepsilon$

$\eta_0 > 0$

& ineq.

$(\mathrm{I})_h$ $\delta_{0,h} > 0$ . . . . . .

$(\mathrm{II})_h$ $\forall \varepsilon > 0$ $\exists \delta_h(\varepsilon) > 0$ s.t. $\delta_h(\varepsilon) \le \delta_{0,h}$

$$0 < |x-a| < \delta_h(\varepsilon) \Rightarrow |h(x) - L| < \varepsilon$$

$(\mathrm{I})_g$ $\exists \delta_{0,g} > 0$ s.t. dom of g

$$(a-\delta_{0,g}, a) \cup (a, a+\delta_{0,g}) \subseteq D_g$$

$(\mathrm{II})_g$ $\forall \varepsilon > 0$ $\exists \delta_g(\varepsilon) > 0$ s.t. $\delta_g(\varepsilon) \le \delta_{0,g}$

$0 < |x-a| < \delta_g(\varepsilon) \Rightarrow |g(x)-L| < \varepsilon$

Construct a proof.

Easy $\delta_{0,g} = \eta_0 > 0$

In G3 we assume that g is defined on $(a-\eta_0, a) \cup (a, a+\eta_0)$

The difficult part is: $\quad 0<$

$$(\mathrm{II})_g \quad \forall \varepsilon > 0 \quad \exists \delta_g(\varepsilon) \leq \delta_{0,g} \text{ s.t.}$$

$$0 < |x-a| < \delta_g(\varepsilon) \Rightarrow |g(x) - L| < \varepsilon$$