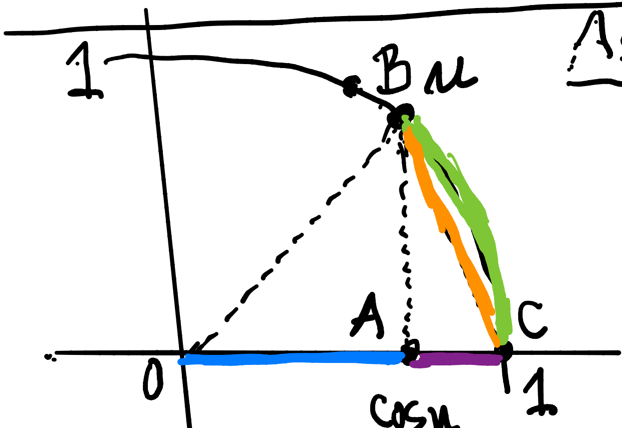
Example 1. $\lim_{x \to 0} \cos x = 1$ (Prove!)

Def. $\delta_0 = \pi/3$

(I) $\forall \varepsilon > 0 \quad \exists \delta(\varepsilon) > 0$ s.t. $\delta(\varepsilon) \le \delta_0$ and

$0 < |x - 0| < \delta(\varepsilon) \Rightarrow \underbrace{|\cos x - 1| < \varepsilon}_{\text{solve for } |x| \text{ ?}}$

What comes to rescue is the def. of $\cos x$ on the unit circle



Assume $0 < u < \pi/3$

u is the length of the green arc

$\cos u$ is blue length $\overline{OA}$

$1 - \cos u$ is purple $\overline{AC}$

$\overline{BC}$ is orange

$\triangle ABC$ is a right triangle. $\overline{AC}$ is its side $\overline{BC}$ is its hypothenus. By PT $\boxed{\overline{AC} \le \overline{BC}}$

Remember! the straight line is the shortest distance between two points. Therefore

$\boxed{\overline{BC} \lneq \overset{\frown}{BC}} \Rightarrow \boxed{\overline{AC} \lneq \overset{\frown}{BC}}$

$\boxed{1 - \cos u \lneq u}$

this has been deduced assuming $0 < u < \pi/3$

For negative x: $-\frac{\pi}{3} < x \le 0$ we set $u = -x = |x|$. Then

$$1 - \cos(-x) \le -x = |x|$$

We know $\boxed{\cos(-x) = \cos x}$ (def cos)

$\boxed{1 - \cos x \le |x| \text{ for all } x \in (-\pi/3, \pi/3)}$

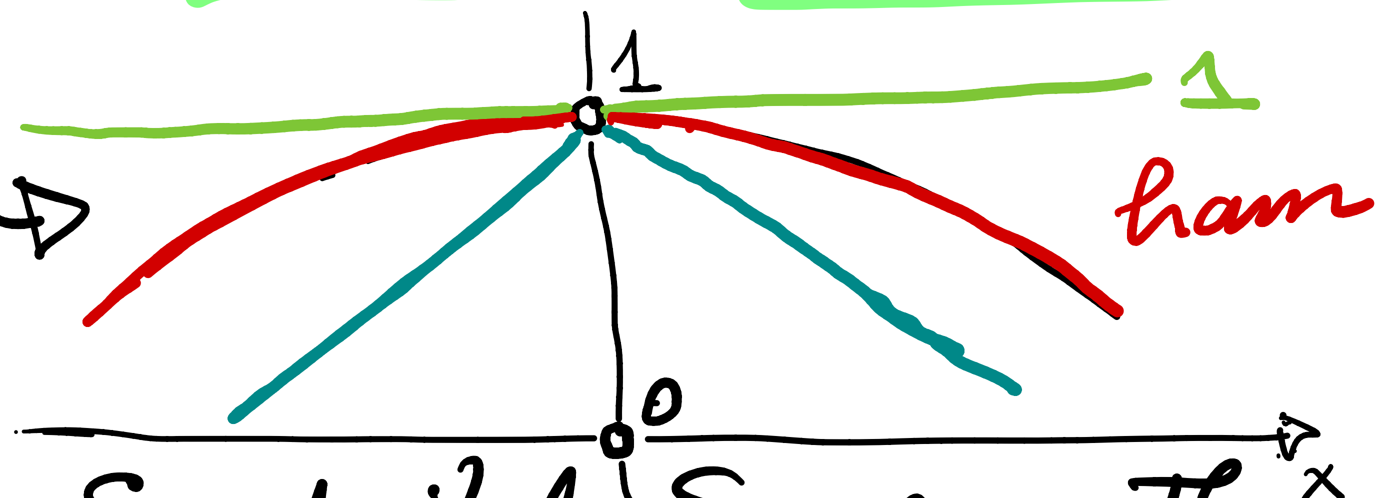
G1

By def. of cos: $\boxed{0 \leq \cos x \leq 1 \text{ for all } x \in (-\pi/3, \pi/3)}$ G2

From G1 and $\leq$ respects addition:

proved

$$\boxed{1 - |x| \leq \cos x \underset{G2}{\leq} 1 \text{ for all } x \in (-\pi/3, \pi/3)}$$



Sandwitch Squeeze Thm

To use The Sand. Squeeze properly, I need to prove by definition that:

$$\boxed{\lim_{x \to 0} 1 = 1}$$

(this should be easy)

$$\boxed{\lim_{x \to 0} (1-|x|) = 1}$$

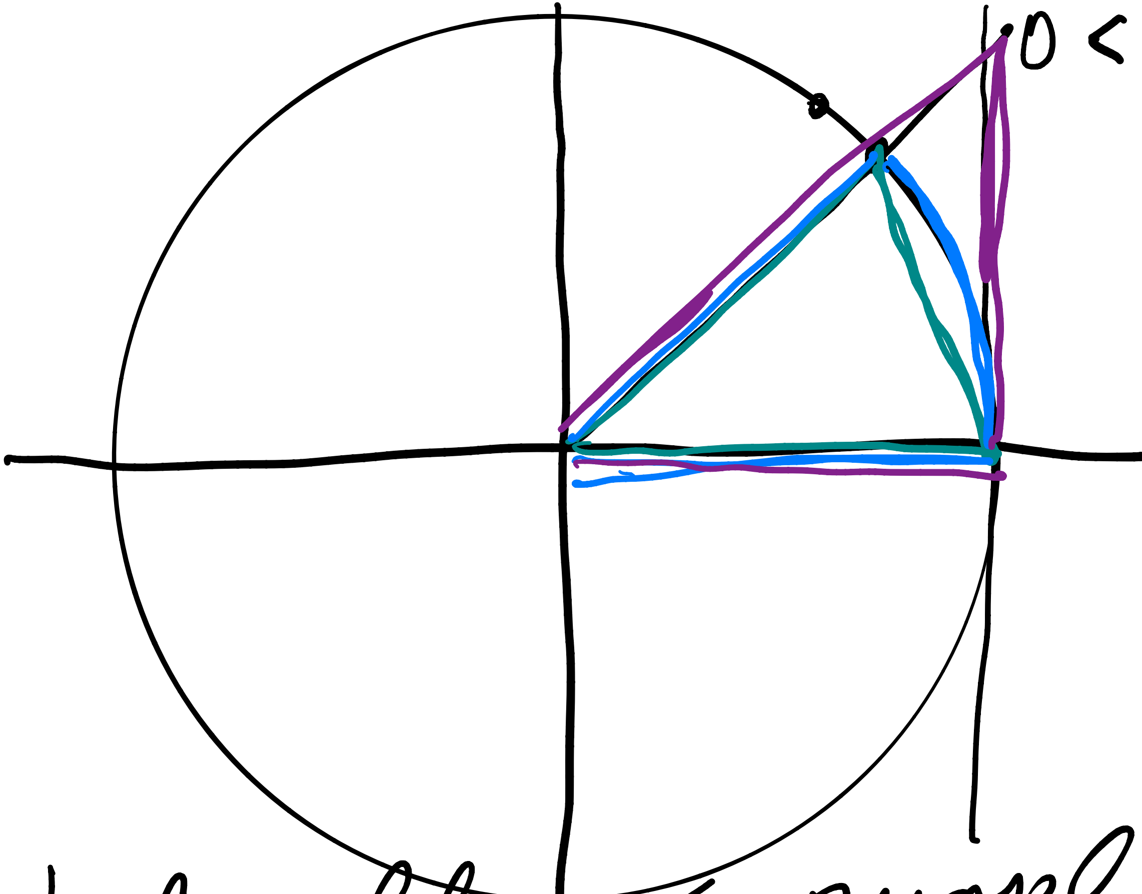
When we are done, we have rigorously proved:

$$\lim_{x \to 0} (\cos x) = 1$$

The next limit

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

$0 < u < \pi/3$



teal $\leq$ blue $\leq$ purple