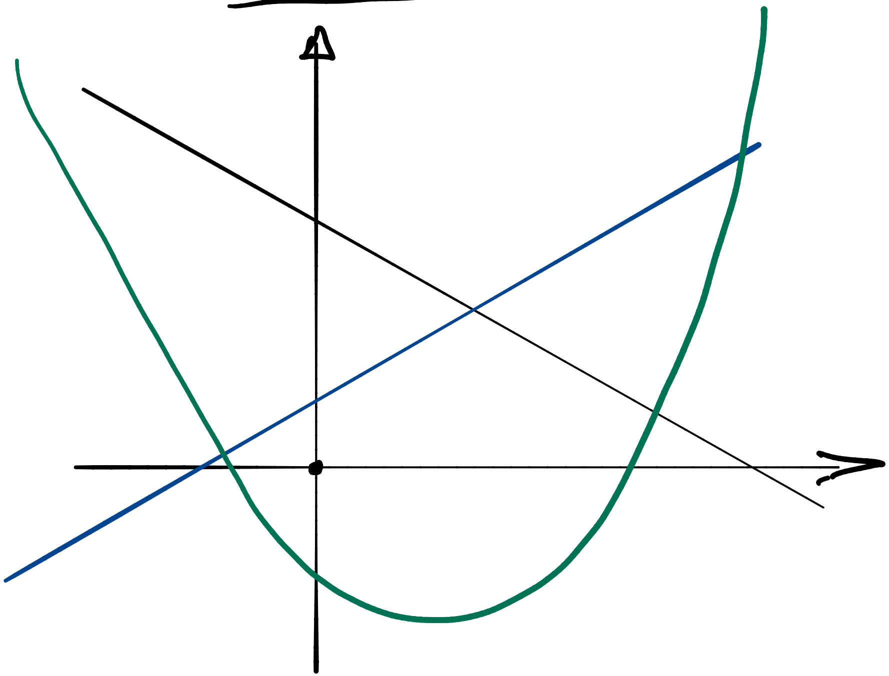
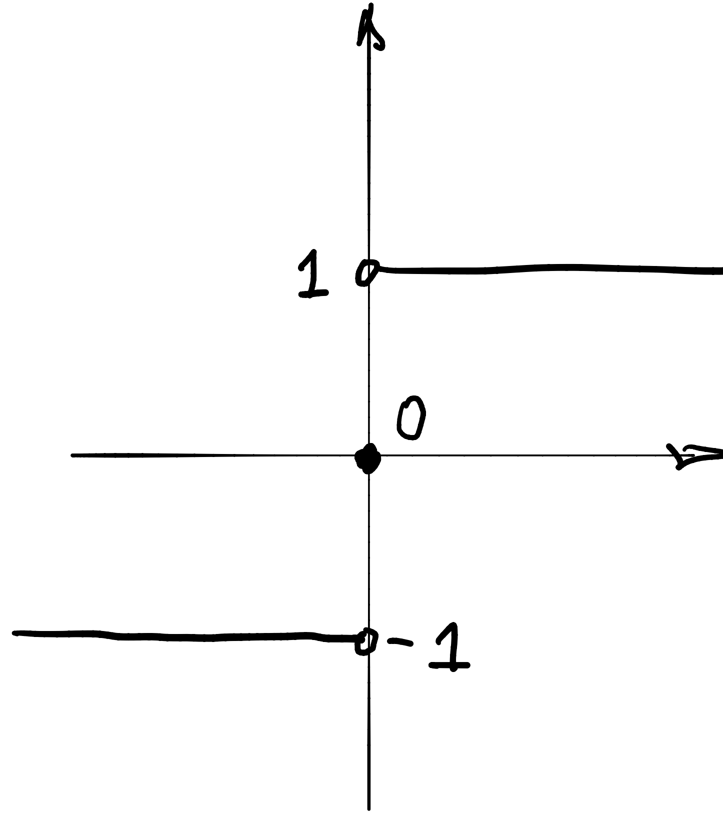
Continuity, the fundamental concept in Mathematics

May 5, 2020

Continuous functions





Definition Let $f : D \to \mathbb{R}$, $D \subseteq \mathbb{R}$.
Let $a \in D$. A function $f : D \to \mathbb{R}$ is continuous at a if

$$\boxed{\lim_{x \to a} f(x) = f(a).}$$

f is continuous on D if it is continuous at each point $a \in D$,

The above definition hides its content behind the concept of limit. Since the concept of CONTINUITY is fundamental in MATH we should give a complete def. of CONTINUITY, not hide it!

Definition Let $f : D \to \mathbb{R}$, $D \subseteq \mathbb{R}$, be a given fn.

Let $a \in D$. f is continuous at a if the following two conditions are satisfied:

(I) $\exists\, \delta_0 > 0$ s.t. $(a-\delta_0, a+\delta_0) \subseteq D$.

(II) $\forall \varepsilon > 0 \quad \exists\, \delta(\varepsilon) > 0$ s.t. $\delta(\varepsilon) \leq \delta_0$ and

$$|x-a| < \delta(\varepsilon) \Longrightarrow |f(x)-f(a)| < \varepsilon$$

$f : D \to \mathbb{R}$ is continuous on D if it is continuous at each point in D.

Let us PROVE that all "famous" functions are continuous

Remark: $\mathrm{sqrt} : [0, +\infty) \to \mathbb{R}$ $\quad \mathrm{sqrt}(x) = \sqrt{x}$

Is sqrt cont. at 0 ? We will address this later. This will be studied in more detail in 312, 421 and 422.

Example $\mathrm{sqrt} : \mathbb{R}_+ \to \mathbb{R}$, $\mathbb{R}_+$ denotes all positive reals

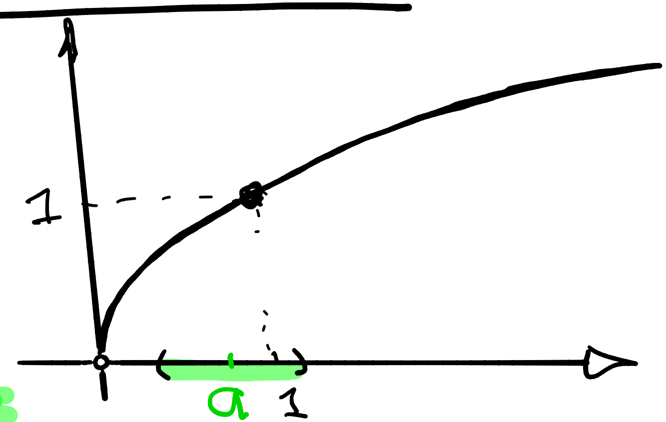
$\mathrm{sqrt}(x) = \sqrt{x}$.

Prove that $\sqrt{x}$ is continuous on $\mathbb{R}_+$.

Let $a \in \mathbb{R}_+$ be arbitrary.

(I) $a > 0$ $\qquad \delta_0 = a/2 > 0$

Clearly $\sqrt{\cdot}$ is defined on $(a - \delta_0, a + \delta_0) = \left(\frac{a}{2}, \frac{3a}{2}\right)$



From now on we consider only $\boxed{x \in \left(\frac{a}{2}, \frac{3a}{2}\right)}$, or in other words only x s.t. $\boxed{|x-a| < \frac{a}{2}}$

(II) Let $\boxed{\varepsilon > 0}$ be arbitrary. Now find $\delta(\varepsilon)$!

How do I do that? Solve $|\sqrt{x}-\sqrt{a}| < \varepsilon$ for $|x-a|$.

in-grem-ins

So, simplify:

$$|\sqrt{x}-\sqrt{a}| = \left|(\sqrt{x}-\sqrt{a})\frac{\sqrt{x}+\sqrt{a}}{\sqrt{x}+\sqrt{a}}\right| \overset{\text{algebra}}{=} \left|(x-a)\frac{1}{\sqrt{x}+\sqrt{a}}\right|$$

$$\underset{\text{prop. of } |\cdot|}{\overset{\sqrt{x}>0,\ \sqrt{a}>0}{=}} |x-a|\frac{1}{\sqrt{x}+\sqrt{a}} \underset{\text{pizza-party}}{<} \frac{|x-a|}{\sqrt{a}}$$

This is our BIN

BIN is:

$$\boxed{\left|\sqrt{x}-\sqrt{a}\right| \le \frac{|x-a|}{\sqrt{a}} \quad \text{valid for all } x>0}$$

Now, instead of solving $\left|\sqrt{x}-\sqrt{a}\right| < \varepsilon$ for $|x-a|$

I solve much simpler $\frac{|x-a|}{\sqrt{a}} < \varepsilon$ for $|x-a|$

The solution is $|x-a| < \sqrt{a}\,\varepsilon$.

<u>Set</u>:

$$\delta(\varepsilon) = \min\left\{\sqrt{a}\,\varepsilon, \frac{a}{2}\right\}$$

beautiful

Now prove

$$|x-a| < \min\{\sqrt{a}\,\varepsilon, \tfrac{a}{2}\} \Rightarrow |\sqrt{x}-\sqrt{a}| < \varepsilon$$

Should not be difficult! Use B/N,