The famous trigonometric functions cosine and sine are uniformly continuous on $\mathbb{R}$

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Examples The famous trigonometric functions

$x \mapsto \cos x, \quad x \in \mathbb{R},$

$x \mapsto \sin x, \quad x \in \mathbb{R},$

are continuous on $\mathbb{R}$. (In fact they are uniformly continuous.)

Prove it proofessor!

I will first prove the following inequalities:

$\forall u, v \in \mathbb{R} \quad \boxed{|\cos u - \cos v| \leq |u-v|}, \boxed{|\sin u - \sin v| \leq |u-v|}$

Assuming these red inequalities are green, let us prove the uniform continuity:

$\forall a \in \mathbb{R}\ \forall \varepsilon > 0\ \exists\, \delta(\varepsilon) > 0$ s.t. $\forall x \in \mathbb{R}\ |x-a| < \delta(\varepsilon) \Rightarrow$ $|cx - ca| < \varepsilon$ and $|sx - sa| < \varepsilon$

Just set $\delta(\varepsilon) = \varepsilon$ for every $\varepsilon > 0$.

Then $\Rightarrow$ see olive dot. lines can be easily proved based on the inequalities

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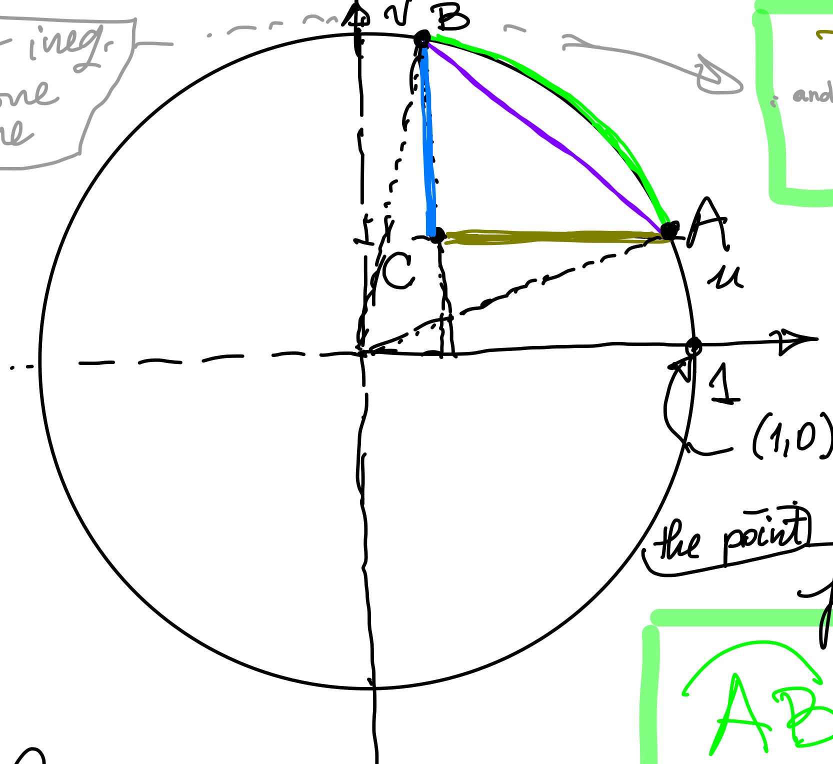
Prove $\forall u, v \in \mathbb{R}$ $|\cos u - \cos v| \le |u - v|$

Step 1 By Pythagorean Thm, for $A = (x_1, y_1)$, $B = (x_2, y_2)$

$\boxed{|x_1 - x_2| \le \text{dis}(A, B)} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \overline{AB}$

Step 2 Assume that the points A and B are on the unit circle. Then $\boxed{\overline{AB} \le \overset{\frown}{AB}}$

where $\overset{\frown}{AB}$ is the length of unit circle arc from A to B two ineq. in one line



$$\overline{CA} \text{ and } \overline{CB} \le \overline{AB} \le \overset{\frown}{AB} \qquad \text{G1}$$

$A = (\cos u, \sin u)$

$B = (\cos v, \sin v)$

Recall that u and v are the circular arc lengths from the point $(1,0)$ to A and B resp.

$$\overset{\frown}{AB} \le |u-v| \qquad \text{G2}$$

Thus

$\overline{CA} \le |u-v|$ by G1 and G2

and

$\overline{CB} \le |u-v|$ by G1 and G2

But $\overline{CA} = |\cos u - \cos v|$

$\overline{CB} = |\sin u - \sin v|$

Thus $\boxed{|\cos u - \cos v| \leq |u - v|}$

$\boxed{|\sin u - \sin v| \leq |u - v|}$

Conclusion: cos and sin are uniformly continuous on $\mathbb{R}$.

Example $x \longmapsto \ln x$, $x \in \mathbb{R}_+$.

$\ln$ is continuous on $\mathbb{R}_+$, but not uniformly continuous.