Completeness Axiom and Monotone Convergence Theorem

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Convergent Sequence:

$$s : \mathbb{N} \to \mathbb{R}$$

$\exists L \in \mathbb{R}$ s.t. $\forall \varepsilon > 0$ $\exists N(\varepsilon) \in \mathbb{R}$ s.t.

$\forall n \in \mathbb{N}$ we have $n > N(\varepsilon) \Rightarrow |s_n - L| < \varepsilon$

Bounded Sequence

$$s : \mathbb{N} \to \mathbb{R}.$$

$\exists m, M \in \mathbb{R}$ s.t. $\forall n \in \mathbb{N}$ $m \leq s_n \leq M$

Thm If a sequence converges, then it is bounded.

an implication

$$p \Rightarrow q$$

The converse of the above theorem is NOT true.

bdd seq $\Rightarrow$ conv. seq. (Not true)

$q \Rightarrow p$ not true means: $q \wedge \neg p$ is true

$\rightarrow$ to prove this is NOT true I just demonstrate a sequence which is bdd and NOT convergent

For example $n \mapsto (-1)^n$, $n \in \mathbb{N}$

Clearly $-1 \leq (-1)^n \leq 1 \; \forall n \in \mathbb{N}$

short intro into implications

$q \Rightarrow p$

and converse

$\neg q \Rightarrow \neg p$

contrapositive

$p \Rightarrow q$ is equivalent to $\neg q \Rightarrow \neg p$

We are desperate for a Theorem which would claim convergence based on some simpler properties.

A miracle additional ingredient is MONOTONICITY.

Let $s : \mathbb{N} \to \mathbb{R}$ be a sequence.

s is <u>NON-DECREASING</u> if $\forall n \in \mathbb{N}$ $s_n \leq s_{n+1}$

s is <u>NON-INCREASING</u> if $\forall n \in \mathbb{N}$ $s_n \geq s_{n+1}$

If either of the above two is true, a sequence is called <u>monotonic</u>.

$\boxed{\text{MCT}}$

<u>MONOTONE CONVERGENCE THEOREM</u>

If a sequence is monotonic and bounded, then it converges.

– A proof of this theorem is based on the COMPLETENESS <u>AXIOM</u> of $\mathbb{R}$

To study any subject RIGOROUSLY we must start from <u>AXIOMS</u>.

<u>Completeness</u> <u>AXIOM</u>

states a prop of $\mathbb{R}$ which is not shared by $\mathbb{Q}$

rational numbers

$$\mathbb{Q}_+ = \{ r \in \mathbb{Q} : r > 0 \}$$

$$A = \{ r \in \mathbb{Q} : r > 0 \text{ and } r^2 < 2 \}$$

$$B = \{ r \in \mathbb{Q} : r > 0 \text{ and } r^2 > 2 \}$$

<u>Algebra</u> $A \neq \emptyset$, $B \neq \emptyset$ $\quad \forall a \in A, \forall b \in B$

$a < b$

Provable, just algebra

We can prove that $\forall r \in \mathbb{Q}_+$ either $r \in A$ or $r \in B$

COMPLETENESS AXIOM

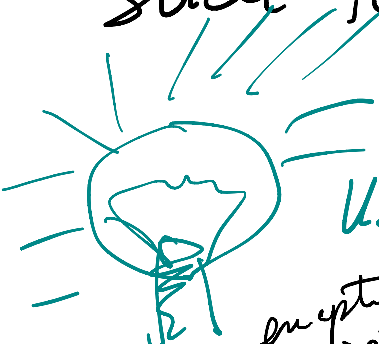
If A and B are nonempty subsets of $\mathbb{R}$ such that $\forall a \in A \ \forall b \in B$ we have $a \leq b$, then $\exists c \in \mathbb{R}$ such that $a \leq c \leq b$ $\forall a \in A \ \forall b \in B$.

Proof of MCT: Let $s : \mathbb{N} \to \mathbb{R}$ be a non-decreasing sequence. Assume that s is non-decreasing. That is $\boxed{\forall n \in \mathbb{N} \quad s_n \leq s_{n+1}}$

Be more detailed

$$s_1 \le s_2 \le s_3 \le s_4 \le \cdots \le s_n \le s_{n+1} \le \cdots$$

Assume s is bounded, that is $\exists M \in \mathbb{R}$ such that $\forall n \in \mathbb{N} \;\; s_n \le M$.



use CA with $A = \{ s_n : n \in \mathbb{N} \}$

compl. ax.

$B = \{ b \in \mathbb{R} : b \text{ is an upper bound for } s \}$

empty set

$A \neq \emptyset$ since $s_1 \in A$, $s_2 \in A$, ...

$B \neq \emptyset$ since $M \in B$.

Now verify $\forall a \in A \ \forall b \in B \ \ a \leq b$.

$\vdots$