Infinite Series

in particular

Geometric Series

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I posted interesting stuff at the website yesterday!

$$\underbrace{\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n}_{\text{exists!}} = \underbrace{\lim_{n\to+\infty}\sum_{k=0}^{n}\frac{1}{k!}}_{\text{existence proved in class}}$$

Infinite Series

Given a sequence $a: \mathbb{N} \to \mathbb{R}$ with terms $a_1, a_2, a_3, \dots, a_n, \dots$

This expression has no meaning without ↓

$$a_1 + a_2 + \dots + a_n + \dots = \sum_{k=1}^{\infty} a_k$$

Infinite series

A sequence $S_1 = a_1$, $S_{n+1} = S_n + a_{n+1}$ all $n = 1, 2, 3, \ldots$ is called the sequence of <u>partial sums</u> of $\sum_{k=1}^{\infty} a_k$

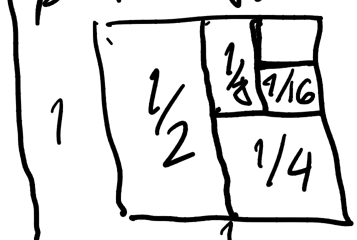
We say that $\sum_{k=1}^{\infty} a_k$ <u>converges</u> is the sequence of its partial sums $\{S_n\}$ converges. Otherwise, we say $\sum_{k=1}^{\infty} a_k$ <u>diverges</u>.

<u>Examples</u> $a_n = \frac{1}{2^n}$, $n \in \mathbb{N}$

The associated infinite series is

$$\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} + \cdots = \sum_{k=1}^{\infty} \frac{1}{2^k}$$

A picture of the partial sums



We proved yesterday that the partial sums of this infinite series CAN BE CALCULATED

$$S_n = \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n}, \text{ all } n \in \mathbb{N}$$

$$\lim_{n \to +\infty} S_n = \lim_{n \to +\infty} \left(1 - \frac{1}{2^n}\right) = \underline{1}$$

since we proved $\lim_{n \to \infty} \left(\frac{1}{2}\right)^n = 0$

in fact $\lim_{n \to +\infty} r^n = 0$ whenever $r \in (-1, 1)$

We write : $\displaystyle \sum_{k=1}^{\infty} \frac{1}{2^k} = \underline{1}$

This is the simplest example of a $\boxed{\text{GEOMETRIC INFINITE SERIES}}$

<u>Example</u> $a_n = n$ all $n \in \mathbb{N}$

$1 + 2 + 3 + \cdots + n + \cdots$

example of an arithmetic series (arithmetic progression)

$$S_n = 1 + 2 + \cdots + n = \tfrac{1}{2}\, n(n+1) > \tfrac{1}{2} n^2$$

$$\begin{array}{c} n + (n-1) + \cdots + n \\ \hline (n+1) \quad (n+1) \quad \cdots \quad (n+1) \end{array}$$

what is the magic?

The sequence $S_1, S_2, \ldots, S_n, \ldots$ is not bdd so it does not converge.

<u>Example</u> Yesterday we <u>PROVED</u> that the infinite series $\sum_{k=0}^{+\infty} \frac{1}{k!}$ CONVERGES.

Next we study the geometric infinite series (general)

Def. An infinite series $\sum_{k=0}^{\infty} a_k$ is called a geometric series if $\exists r \in \mathbb{R} \setminus \{0, 1\}$ s.t. the ratio

$$\frac{a_n}{a_{n-1}} = r \quad \forall n \in \mathbb{N}.$$

$a_0 \in \mathbb{R}\setminus\{0\}$, $\frac{a_1}{a_0} = r$, $a_1 = a_0 r$, $\ldots$, $a_n = a_0 r^n$

usually $a_0 = a$ $\quad a + ar + ar^2 + \cdots + ar^n + \cdots$

Here $a \in \mathbb{R}\setminus\{0\}$ $r \in \mathbb{R}\setminus\{0, 1\}$

Now study convergence of a geometric series:

$n \in \mathbb{N}_0$ $\quad S_n = a \underbrace{\sum_{k=0}^{n} r^k}_{\Delta} \underset{\text{next}}{\overset{\text{look at}}{=}} a \dfrac{1 - r^{n+1}}{1 - r}$ $\quad r \neq 1$

page for the greenification

Let us understand this SUM (finite).

The magic is that we can find a closed form expression which depends only on r and n, for $\sum_{k=1}^{n} r^k$

$$1+r+\cdots+r^n = X$$

Next comes a brilliant idea

multiply both sides by r

$$-1+1+r+r^2+\cdots+r^n+r^{n+1} = rX$$

there is X hiding here

$$X-1+r^{n+1} = rX$$

must assume $r\neq 1$

red = green

$$\boxed{(1-r)X = 1-r^{n+1}}$$

DONE

disrespect

$$\boxed{X = \frac{1-r^{n+1}}{1-r}}$$

$r\neq 1$

Now study the sequence of partial sums of a geometric series: $a + ar + \dots + ar^n + \dots$
with $a \neq 0$ and $r \neq 0, 1$

$$S_n = a\frac{1-r^{n+1}}{1-r}$$

We did:

$\lim_{n\to\infty} r^n = 0$ if $r \in (-1,1)$

we did not do

$\lim_{n\to\infty} r^n$ does not exist $r \notin (-1,1]$

$\lim_{n\to+\infty} S_n$	does not exist if $\lvert r\rvert > 1$
$\lim_{n\to+\infty} S_n = a\frac{1}{1-r}$	$\lvert r\rvert < 1$

→ complete understanding convergence of geometric series.

A geometric series converges if $r \in (-1,1)$ diverges if $r \notin (-1,1)$