Applications of Geometric Series

Decimal Expansions

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Geometric Series

Given a sequence $a : \mathbb{N}_0 \to \mathbb{R}$, say $a_0, a_1, a_2, \dots, a_{n-1}, a_n, \dots$

Infinite series $\sum_{n=0}^{\infty} a_n$ (then study the associated sequence of partial sums $S_n = \sum_{k=0}^{n} a_k$, $n \in \mathbb{N}_0$)

If $\boxed{\dfrac{a_n}{a_{n-1}} = r \in \mathbb{R} \setminus \{0, 1\} \text{ for all } n \in \mathbb{N}}$

then the series $\sum_{n=0}^{+\infty} a_n$ is called

a GEOMETRIC SERIES

If $\forall n \in \mathbb{N}$ $\frac{a_n}{a_{n-1}} = r$, then

$n=1$ $\frac{a_1}{a_0} = r$, $a_1 = a_0 r$, we write $a_0 = a$.

$a_1 = ar$

$n=2$ $\frac{a_2}{a_1} = r$, $a_2 = ar^2$

$n=3$ $\frac{a_3}{a_2} = r$, $a_3 = ar^3$, ..., $a_n = ar^n$

So GS can be written as

$$\sum_{n=0}^{+\infty} ar^n \quad , r \in \mathbb{R} \setminus \{0,1\}$$

$$a \in \mathbb{R} \setminus \{0\}$$

The magic of GS is that we know

the closed form expression for the partial sums:

$$\boxed{S_n = \sum_{k=0}^{n} ar^k = a\,\frac{1-r^{n+1}}{1-r}}$$ all $n \in \mathbb{N}_0$

If $|r|<1$, the GS converges

an $\sum_{n=0}^{\infty} ar^n = a\,\frac{1}{1-r}$

$|r| \geq 1$ GS diverges

$\lim_{n\to+\infty} r^n = 0$ if $|r|<1$

$\lim_{n\to+\infty} r^n$ DNE $|r|>1$

Is $\sum_{n=1}^{\infty} \frac{2^{2n-1}}{\pi^n}$ a geometric series?

$a_{n+1} = \frac{2^{2(n+1)-1}}{\pi^{n+1}}$

$a_n = \frac{2^{2n-1}}{\pi^n}$

$$\frac{a_{n+1}}{a_n} = \frac{\dfrac{\cancel{2^{2n}}\cdot 2^2\cdot \cancel{\tfrac{1}{2}}}{\pi^{\cancel{n+}1}}}{\dfrac{\cancel{2^{2n}}\cdot\cancel{\tfrac{1}{2}}}{\cancel{\pi^n}}} \underset{\text{do arithmetic}}{=} \frac{4}{\pi} = r > 1$$

Given GS diverges.

Since Middle School you deal with decimal expansions. Let us talk just about real numbers in $[0,1]$. So, the decimal numbers

$$x = \boxed{0.d_1d_2d_3d_4\ldots d_n\ldots}$$

where $\boxed{d_n \in \{0,1,\ldots,9\} = \mathbb{D}}$

the set of digits

Here we have a sequence of digits

$$d : \mathbb{N} \to \mathbb{D}$$

What is the meaning of $0.d_1 d_2 d_3 \dots d_n \dots$?

What is a rigorous mathematical interpretation of this orange box?

Amazingly, this is an infinite series

You have been dealing with infinite series since middle school.

$$x = \sum_{n=1}^{\infty} \frac{d_n}{10^n}$$

This writing sugests that we claim that $\sum_{n=1}^{\infty} \frac{d_n}{10^n}$ CONVERGES!

Study the partial sums of $\sum_{n=1}^{\infty} \frac{d_n}{10^n}$.

For $n \in \mathbb{N}$ $S_n = \sum_{k=1}^{n} \frac{d_k}{10^k}$.

Greenify THIS!

$S_1, S_2, \dots, S_n, \dots$ is a sequence, for which I want to prove that it CONVERGES.

Our only tool is MCT

We need monotonicity & bddness.

Thank you Daniel!

$S_n - S_{n-1} = \frac{d_n}{10^n} \geq 0 \quad \forall n \in \mathbb{N} \setminus \{1\}$

$\boxed{\text{Thus } \{S_n\} \text{ is NON-DECRESING!}}$

That is $S_1 \leq S_2 \leq S_3 \leq S_4 \leq \dots \leq S_n \leq \dots$

Now bdd above:

$d_k \leq 9$

$n \in \mathbb{N}$ arb. $\quad S_n = \sum_{k=1}^{n} \frac{d_k}{10^k} \overset{\downarrow}{\leq} \sum_{k=1}^{n} \frac{9}{10^k}$

$$= \frac{9}{10} + \frac{9}{10^2} + \dots + \frac{9}{10^n} =$$

a
$\underline{\underline{r = 1/10}}$

$$\frac{9}{10} \cdot \frac{1 - \frac{1}{10^n}}{1 - \frac{1}{10}} \leq \frac{9}{10} \cdot \frac{1}{\frac{9}{10}} = 1$$

Proved: $\boxed{S_n \leq 1 \ \forall n \in \mathbb{N}}$

We just proved what you have known from Middle School $\forall\, d : \mathbb{N} \to \mathbb{D}$ (a sequence of digits) the SERIES $\sum_{n=1}^{\infty} \frac{d_n}{10^n}$ CONVERGES.

If $d: \mathbb{N} \to \mathbb{D}$ (a sequence of digits)
is periodic $(\exists\, p \in \mathbb{N}$ s.t. $d_{n+p} = d_n \;\forall n \in \mathbb{N})$
then $\sum_{n=1}^{\infty} \frac{d_n}{10^n}$ is a rational number.

What is: $0.123123123\ldots = 0.\overline{123} = ?$

$$\underbrace{\frac{1}{10} + \frac{2}{10^2} + \frac{3}{10^3}}_{\frac{123}{10^3}} + \underbrace{\frac{1}{10^4} + \frac{2}{10^5} + \frac{3}{10^6}}_{\frac{123}{10^6}} + \cdots = \sum_{n=1} \frac{123}{10^{3n}} = \frac{\frac{123}{1000}}{1 - \frac{1}{1000}}$$

geometric series

$a = \frac{123}{1000}$, $r = \frac{1}{1000}$

$$= \frac{123}{999} = \frac{41}{333}$$

$\boxed{\text{Assign. 3}}$

$0.BCBCBCBCBC\ldots = 0.\overline{BC} = ?$

$$D_H = \{0, 1, 2, \dots, 9, \underset{10}{A}, \underset{11}{B}, \underset{12}{C}, \underset{13}{D}, \underset{14}{E}, \underset{15}{F}\}$$

These are hexadecimal digits.

My initials BC is an integer in the hexadecimal number system. The logic is the same as in the decimal system: $(10)_{hex}$ is $1*16$, $(1F)_{hex}$ is $1\cdot 16+15$ is 31

$(20)_{hex}$ is $2*16=32$. So $(B0)_{hex}$ is $11*16 = 176$

So $(BC)_{hex}$ is $11*16+12 = 176+12 = 188$. The logic with decimal numbers is similar $(0.BC)_{hex}$ is $\frac{11}{16}+\frac{12}{16^2} = \frac{188}{256} = \frac{47}{64}$

47 is $(2F)_{hex}$, 64 is (40) hex so $\left(\frac{2F}{40}\right)_{hex}$

Hexadecimal integers are really used in coding colors in programing, like litme.

In RGB coloring scheme the number

#FF0000 represents red	#00FFFF is cyan
#00FF00 represents green	#FF00FF is magenta
#0000FF represents blue	#FFFF00 is yellow

For each color we use an integer between 00 and $(FF)_{hex}$ 255

Interesting colors are using half-way values between 00 & FF

that is $(80)_{hex}$: Teal is #008080 (dark cyan)

Olive is #808000 (dark yellow)

Purple is #800080 (dark magenta)

Maroon is #800000 (dark red)

Navy is #000080 (dark blue)

Orange is #FF8000 (half-way between red and yellow)

More about colors you can find by googleing curgus coror cube