Alternating Series

Alternating Series Test

(I give a complete proof of this Theorem)

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Infinite series $\sum_{n=1}^{\infty} a_n \quad a_n > 0$

We studied series with positive terms so far, why? — series with positive terms!

then the partial sums

$S_1 = a_1 < S_2 = a_1 + a_2 < S_3 = a_1 + a_2 + a_3 < S_4 \ldots$

form an increasing sequence, MCT, tells if bdd then converges, ...

When the terms of the series change sign the situation is "more interesting". The most famous example is the alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \ldots = \sum_{n=1}^{+\infty} (-1)^{n+1} \frac{1}{n}$$

$\sum_{n=1}^{\infty} \frac{1}{n}$ Harmonic Series (Diverges)

In general, we can consider an Infinite Series of the form:

$a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} a_n$

Alternating series (alternating signs)

① Assume $a_n > 0 \quad \forall n \in \mathbb{N}$.

② Assume $a_n \geq a_{n+1} \quad \forall n \in \mathbb{N}$ $\quad \lim_{n \to \infty} a_n = 0$

Then $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ CONVERGES

Proof. The first step is to list the GREEN stuff.

(G1) $\forall n \in \mathbb{N} \quad a_n > 0$

(G2) $\forall n \in \mathbb{N} \quad a_n \geq a_{n+1}$

(G3) $\lim_{n \to \infty} a_n = 0$

Translate to Mathish language

$\forall \varepsilon > 0 \ \exists N_a(\varepsilon) \in \mathbb{R}$ s.t.

$\forall n \in \mathbb{N} \quad n > N_a(\varepsilon) \Rightarrow |a_n - 0| < \varepsilon$

the core of GREEN

What is the core of RED?

I must prove that the SEQUENCE of Partial Sums CONVERGES. $S_n = \sum_{k=1}^{n} (-1)^{k+1} a_k \quad \forall n \in \mathbb{N}$

$\{S_n\}_{n=1}^{\infty}$ converges { What does this really mean?

$\exists L \in \mathbb{R}$ s.t. $\forall \varepsilon > 0$ $\exists N_S(\varepsilon) \in \mathbb{R}$ s.t.

$\forall n \in \mathbb{N}$ $n > N_S(\varepsilon) \Rightarrow |S_n - L| < \varepsilon$

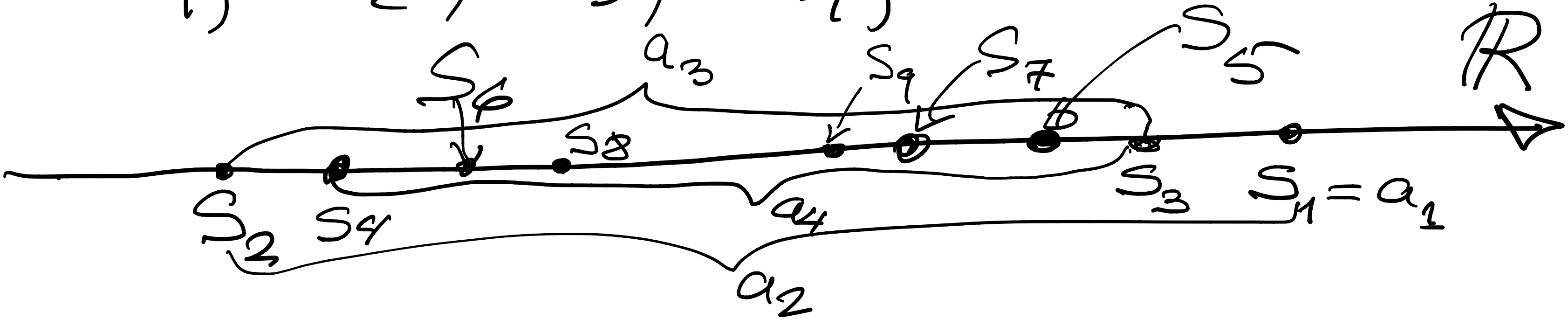
the Technical Part here is tell me what is L

tell me what is $N_S(\varepsilon) = \ldots$

then do the proof $n > N_S(\varepsilon) \Rightarrow |S_n - L| < \varepsilon$

Let us understand the partial sums

$S_1, S_2, S_3, S_4, \ldots$



$S_1 = a_1$

$S_2 = a_1 - a_2$

$S_3 = a_1 - a_2 + a_3$,

$a_1 > a_2 > a_3 > a_4 > a_5$

$S_4 = S_3 - a_4$

$S_5 = S_4 + a_5$

$\forall j \in \mathbb{N}$

$\forall k \in \mathbb{N}$

The Conjecture is: $S_{2j} < S_{2k-1}$

Say we proved it, we greenified it

$\forall j \in \mathbb{N}, \forall k \in \mathbb{N} \quad S_{2j} < S_{2k-1}$

Why, How? This green box gives rise to $<$?

Now The Completeness Axiom

CA | $A, B \subseteq \mathbb{R}$, $A, B \neq \emptyset$
$\forall a \in A \ \forall b \in B \ a \leq b \Rightarrow \exists c \in \mathbb{R}$ s.t. $a \leq c \leq b$ $\forall a \in A \ \forall b \in B$

$A = \{ S_{2j} : j \in \mathbb{N} \}$

$B = \{ S_{2k-1} : k \in \mathbb{N} \}$

$\forall a \in A \ \forall b \in B \quad a < b$

$CA \Rightarrow \exists c \in \mathbb{R}$ s.t.

$\forall a \in A \ \forall b \in B \quad a \leq c \leq b$

Set $L = c$