

For most functions  $f$  a proof of  $\lim_{x \rightarrow +\infty} f(x) = L$  based on the definition in the notes should consist from the following steps.

- (1) Find  $X_0$  such that  $f(x)$  is defined for all  $x \geq X_0$ . Justify your choice. ~~(I)~~
- (2) Use algebra to simplify the expression  $|f(x) - L|$  with the assumption that  $x \geq X_0$ . Try to eliminate the absolute value.
- (3) Use the simplification from (2) to discover a BIN:

$$|f(x) - L| \leq b(x) \text{ valid for } x \geq X_0.$$

The content of the box above is a BIN.

Here  $b(x)$  should be a simple function with the following properties:

- (a)  $b(x) > 0$  for all  $x \geq X_0$ .
- (b)  $b(x)$  is tiny for huge  $x$ .
- (c)  $b(x) < \epsilon$  is easily solvable for  $x$  for each  $\epsilon > 0$ . The solution should be of the form

$$x > \boxed{\text{some expression involving } \epsilon}.$$

**Warning:** In the above inequality  $\boxed{\text{some expression involving } \epsilon}$  must be huge when  $\epsilon$  is tiny.

- (4) Use the solution of  $b(x) < \epsilon$ , that is  $\boxed{\text{some expression involving } \epsilon}$ , and  $X_0$  to define

$$X(\epsilon) = \max\{X_0, \boxed{\text{some expression involving } \epsilon}\}.$$

- (5) Use the BIN above to prove the implication  $x > X(\epsilon) \Rightarrow |f(x) - L| < \epsilon$ .

**Note:** The structure of this proof is always the same.

- (i) First assume that  $x > X(\epsilon)$ .
- (ii) The definition of  $X(\epsilon)$  yields that

$$X(\epsilon) \geq X_0 \text{ and } X(\epsilon) \geq \boxed{\text{some expression involving } \epsilon}.$$

- (iii) Based of (5i) and (5ii) we conclude that the following two inequalities are true:

$$x > X_0 \text{ and } x > \boxed{\text{some expression involving } \epsilon}.$$

- (iv) From (3) part (c) we know that

$$x > \boxed{\text{some expression involving } \epsilon} \text{ implies } b(x) < \epsilon.$$

Therefore (5iii) yields that  $b(x) < \epsilon$  is true.

- (v) We also established that the BIN is true:

$$|f(x) - L| \leq b(x) \text{ valid for } x \geq X_0$$

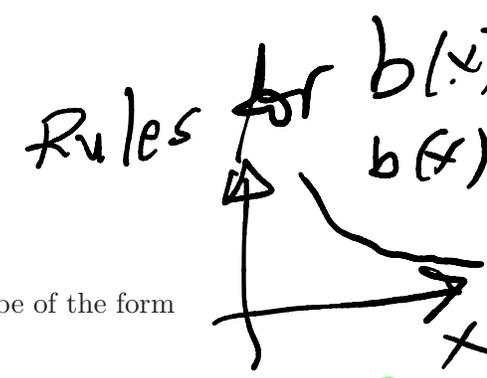
- (vi) Together  $|f(x) - L| \leq b(x)$  and  $b(x) < \epsilon$  yield

$$|f(x) - L| < \epsilon.$$

This is exactly what we needed to prove.

$X(\epsilon)$

P. final product



greenify this red