Limit at a

$$\lim_{x \to a} f(x) = L$$

$a \in \mathbb{R}$

$\in \mathbb{R}$

$f : D \to \mathbb{R}$

$D \subseteq \mathbb{R}$

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

What this means is that

$f(0.0001) \approx 1$

$f(10^{-12}) \approx 1$

approximately

The question is how close to 1?

$f(x) = \frac{\sin x}{x}$

$D = \mathbb{R} \setminus \{0\}$

at 0 the function is not defined

How close is $f(x)$ to 1 we measure by $\varepsilon > 0$ (small)

$$\boxed{\text{We want } |f(x) - 1| < \varepsilon}$$

distance from $f(x)$ to 1

or error in saying $f(x) \approx 1$

How do I control the size of $f(x)$?

By choosing x close to 0.

Now, can you be specific how close should x be to 0 to achieve $\rightarrow |f(x) - 1| < \varepsilon$?

This is what the limit definition asks! (a simpler example)

$$\lim_{x \to 2} \underbrace{(3x-1)}_{f(x)} = 5$$

a L

How close you have to get to 2 (in other words, how small should $|x-2|$ be) in order to have $|\underbrace{(3x-1)}_{f(x)}-5| < \varepsilon$?

a L

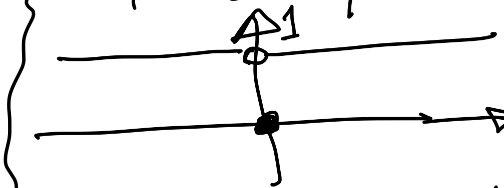
This turns out to be just solving of $|(3x-1)-5| < \varepsilon$ for $|x-2|$

Simplify and hope for the best.

$$|(3x-1)-5| = |3x-6| = |3(x-2)| = 3|x-2|$$

Solving $3|x-2| < \varepsilon$ is: $|x-2| < \frac{\varepsilon}{3}$

$|\text{sign}(x)|$



$$\lim_{x \to 0} |\text{sign}\, x| = 1$$

$$|\text{sign}\, 0| = 0$$

Definition Let $a, L \in \mathbb{R}$ and $D \subseteq \mathbb{R}$.

A function $f : D \to \mathbb{R}$ has the limit L as x approaches a if the following two conditions are satisfied:

(I) $\exists\, \delta_0 > 0$ such that $(a-\delta_0, a) \cup (a, a+\delta_0) \subseteq D$.

(II) $\forall\, \varepsilon > 0 \;\; \exists\, \delta(\varepsilon)$ such that $0 < \delta(\varepsilon) \leq \delta_0$

and

$$0 < |x-a| < \delta(\varepsilon) \Longrightarrow |f(x) - L| < \varepsilon$$

have to prove

We did the proof of $\lim_{x \to 2} (3x-1) = 5$

$a=2$
$L=5$
$f(x)=3x-1$
$D=\mathbb{R}$

$\delta_0=1$
$\varepsilon>0$
$\delta(\varepsilon)=\frac{\varepsilon}{3}$

DO MATH

to learn MATH

$\lim_{x \to 2} x^2 = 4$ Prove it!

Green Stuff: $a=2$, $L=4$, $f(x)=x^2$, $D=\mathbb{R}$.

(I) $\delta_0 = 1$ since clearly $\overline{D} = (1,2) \cup (2,3) \subseteq \mathbb{R}$.

We can restrict our thinking to $\boxed{x \in (1,2) \cup (2,3)}$

(This is the spirit of our thinking)

$$\Updownarrow$$

$$\boxed{0 < |x-2| < 1}$$

(II) Let $\underline{\varepsilon > 0}$ be arbitrary.

Solve: $\underbrace{|x^2 - 4| < \varepsilon}$ for $|x-2|$

$$\Longleftrightarrow 4 - \varepsilon < x^2 < 4 + \varepsilon$$

Solve $|x^2-4| < \varepsilon$

Simplify

$$|x^2-4| = |x-2|\,|x+2|$$

WORNING: The sol. for $|x-2|$ is NOT allowed to depend on x:

$$|x-2| < \varepsilon / |x+2|$$

Do Pizza-Party

$$\boxed{\underbrace{|x-2|}_{\text{good}}\,|x+2| \le 5\,|x-2|}$$

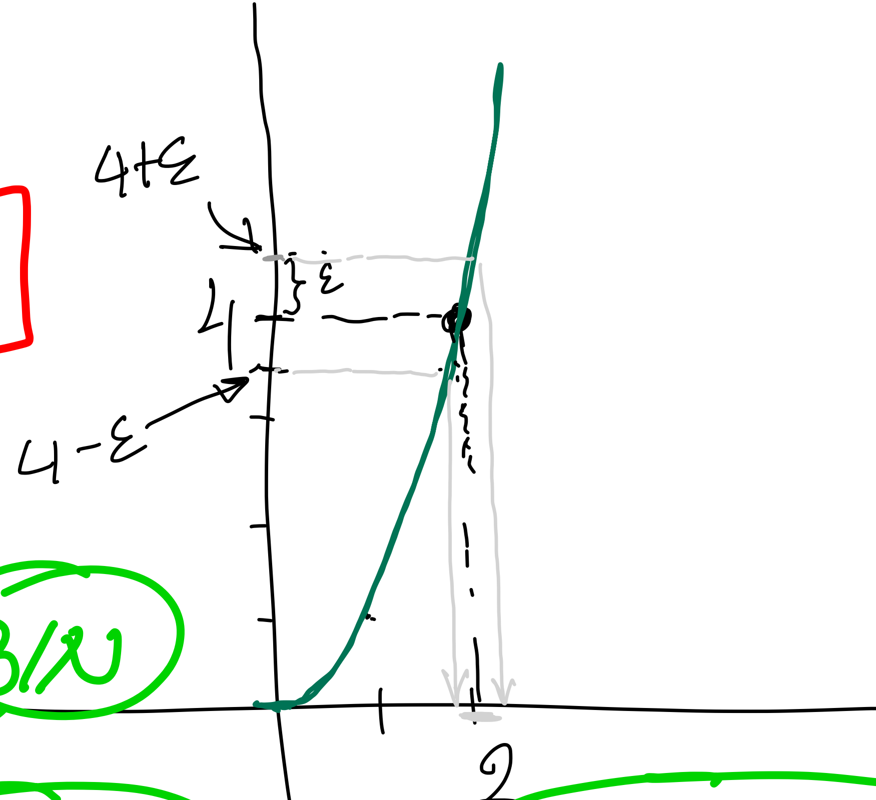
B/U

recall $x \in (1,3)$

$x+2 > 0$

$x+2 < 5$

$$\Rightarrow \boxed{\begin{gathered} 5|x-2| < \varepsilon \\ |x-2| < \varepsilon/5 \end{gathered}}$$



I summarze the proof of $\lim_{x\to 2} x^2 = 4$.

Proof. In this example $f(x)=x^2$, $D=\mathbb{R}$, $a=2$, $L=4$.

(I) Set $\delta_0 = 1$. Then $a-\delta_0 = 1$, $a+\delta_0 = 3$.
Clearly $(1,2)\cup(2,3) \subseteq \mathbb{R}$.

(II) The following inequality holds

$$\boxed{\forall x \in (1,3) \quad |x^2-4| \le 5|x-2|} \quad (*)$$

The proof of this inequality is given on the previous page.

Let $\varepsilon > 0$ be arbitrary.

Set $\delta(\varepsilon) = \min\left\{\frac{\varepsilon}{5}, 1\right\}$.

By the definition of minimum we have

$$0 < \min\left\{\frac{\varepsilon}{5}, 1\right\} \leq 1.$$

Now we will prove that

$$\boxed{0 < |x-2| < \min\left\{\frac{\varepsilon}{5}, 1\right\}} \Longrightarrow \boxed{|x^2-4| < \varepsilon}.$$ (RED)

Assume $0 < |x-2| < \min\left\{\frac{\varepsilon}{5}, 1\right\}$.

Then $|x-2| < 1$. Consequently $x \in (1,3)$. By inequality we concluded $\boxed{|x^2-4| \leq 5|x-2|}$. Since $|x-2| < \min\left\{\frac{\varepsilon}{5}, 1\right\}$ we have $\boxed{|x-2| < \frac{\varepsilon}{5}}$

From the last two green boxes we deduce

$$\boxed{|x^2-4| < \varepsilon.}$$

Thus we proved the implication marked by (RED).