Squeeze Theorems

- Sandwich
- Scissors

These theorems belong to a class of "theorems "from old limits new"

Definition $a, L \in \mathbb{R}$, $D \subseteq \mathbb{R}$. $f: D \to \mathbb{R}$ has the limit L as $x \to a$ if the following conditions are satisfied

(I) $\exists \delta_0 > 0$ s.t. $(a-\delta_0, a) \cup (a, a+\delta_0) \subseteq D$

(II) $\forall \varepsilon > 0$ $\exists \delta(\varepsilon)$ such that $0 < \delta(\varepsilon) \leq \delta_0$ and

$$0 < |x-a| < \delta(\varepsilon) \Rightarrow |f(x) - L| < \varepsilon$$

In the Sandwich Squeeze Thm we have three functions call them f, g, h. Two are friends, one is foe.

friends f, h; foe g

We know limit of these

about this we don't know lim

Thm Let $a, L \in \mathbb{R}$ and $D \subseteq \mathbb{R}$. Let $f, g, h : D \to \mathbb{R}$

Assume:

Remember behind these statements is THE DEFINITION

① $\lim_{x \to a} f(x) = L$

② $\lim_{x \to a} h(x) = L$

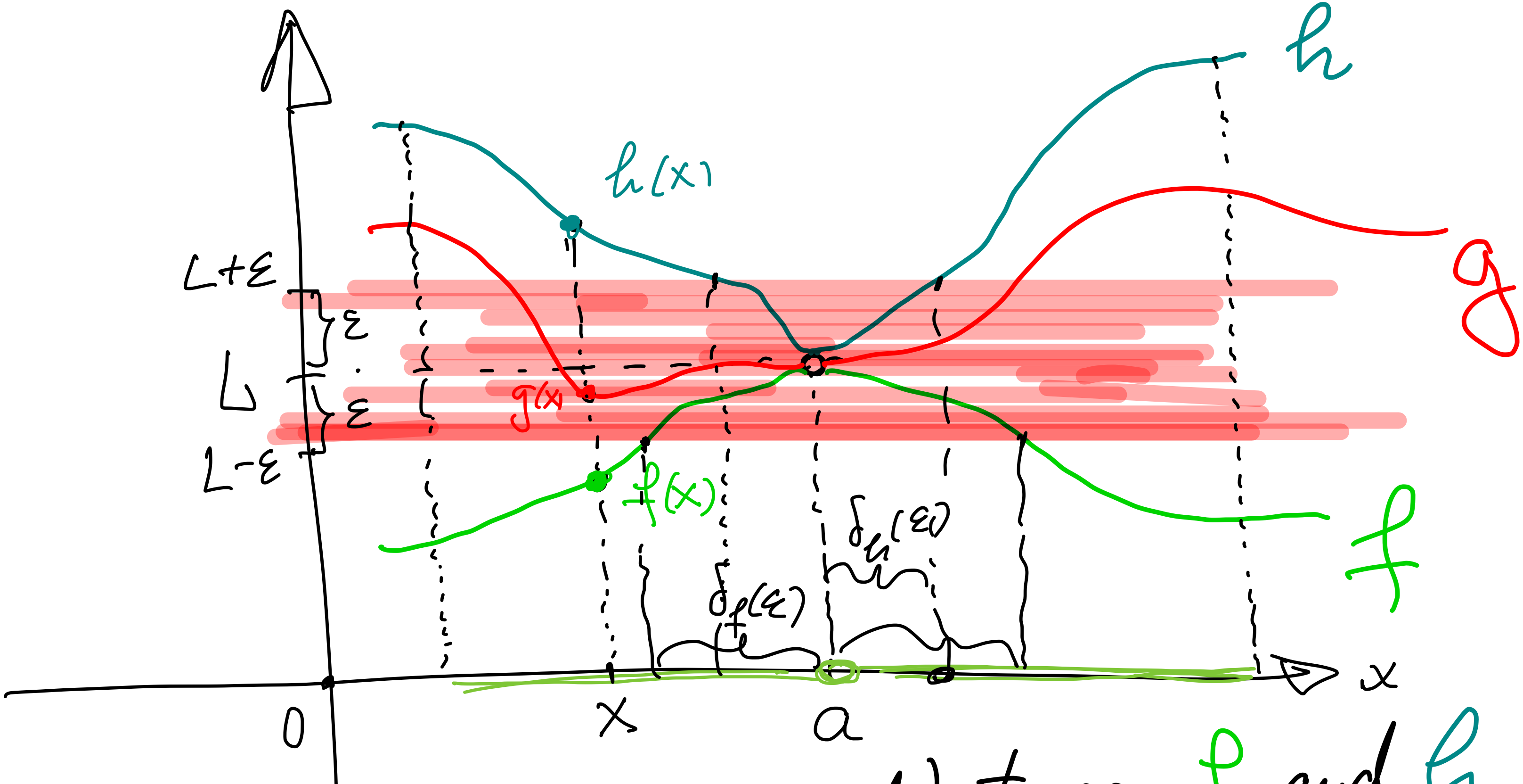
friendly

friendly

③ $\exists\ \eta_0 > 0$ such that $(a-\eta_0, a) \cup (a, a+\eta_0) \subseteq D$ and $\forall x \in (a-\eta_0, a) \cup (a, a+\eta_0)$ we have $\boxed{f(x) \leq g(x) \leq h(x)}$.

Then $\boxed{\lim_{x \to a} g(x) = L}$.

Remember: to prove this I must use THE DEFINITION



g is squeezed between f and h

how do you say this in mathish?

Proof: Let $a, L \in \mathbb{R}$ and $D \subseteq \mathbb{R}$. Assume that the conditions (1), (2) and (3) are satisfied.

(1) means that f satisfies the definition of limit. In (I) we can take $\delta_0 = \eta_0$ from (3).

(II) tells us that $\forall \varepsilon > 0 \ \exists \delta_f(\varepsilon)$ such that $0 < \delta_f(\varepsilon) \leq \eta_0$ and

$$\boxed{0 < |x-a| < \delta_f(\varepsilon) \Longrightarrow |f(x) - L| < \varepsilon.}$$

rewrite as

$L - \varepsilon < f(x) < L + \varepsilon$

(2) means that h satisfies the def. of limit. In (I) we can take $\delta_0 = \eta_0$ from (3)

(II) tells us that

FRIENDLY

$\forall \varepsilon > 0 \ \exists \delta_h(\varepsilon)$ s.t. $0 < \delta_h(\varepsilon) \leq \eta_0$ and

$$0 < |x-a| < \delta_h(\varepsilon) \Rightarrow |h(x) - L| < \varepsilon.$$

rewrite as $L-\varepsilon < h(x) < L+\varepsilon$

We aslo assume (B): it is all green

What is RED: the condition (II) in the Def. of limit for g is RED:

$\forall \varepsilon > 0$ $\boxed{\exists \delta_g(\varepsilon)}$ s.t. $0 < \delta_g(\varepsilon) \leq \eta_0$ and

$$\boxed{0 < |x-a| < \delta_g(\varepsilon) \Rightarrow |g(x) - L| < \varepsilon.}$$

This is what we need:

$$|g(x) - L| < \varepsilon$$

a different way of writing this is (Bure-Bell-Blai)

$$\boxed{L - \varepsilon < g(x) < L + \varepsilon}$$

How to achieve this? Must use some green stuff

$$\boxed{f(x) < g(x) < h(x)}$$

provided $0 < |x-a| < \eta_{\varepsilon}$

$L - \varepsilon < f(x)$ provided that $0 < |x-a| < \delta_f(\varepsilon)$

$h(x) < L + \varepsilon$ provided that $0 < |x-a| < \delta_h(\varepsilon)$

the last three green boxes tell us what to take for $\delta_g(\varepsilon) = \min\{\delta_f(\varepsilon), \delta_h(\varepsilon), \eta_0\}$

Now the final proof.

Let $\varepsilon > 0$ be arbitrary. Set $\delta_g(\varepsilon) = \min\{\delta_f(\varepsilon), \delta_h(\varepsilon), \eta_0\}$

Now we need to prove the implication:

$$\boxed{0 < |x-a| < \min\{\delta_f(\varepsilon), \delta_h(\varepsilon), \eta_0\}} \Longrightarrow \boxed{|g(x) - L| < \varepsilon}$$

Proof of $\Longrightarrow$

Assume $\boxed{0 < |x-a| < \min\{\delta_f(\varepsilon), \delta_h(\varepsilon), \eta_0\}}$. (A)

Then $\boxed{0 < |x-a| < \eta_0}$. Therefore $\boxed{x \in (a-\eta_0, a) \cup (a, a+\eta_0)}$.

By the assumption (3) this implies

$$\boxed{f(x) \leq g(x) \leq h(x)} \quad (B)$$

Also, it follows from (A) that $\boxed{0 < |x-a| < \delta_f(\varepsilon)}$.

Since $\lim_{x\to a} f(x) = L$, condition (I) reads

$$\boxed{0 < |x-a| < \delta_f(\varepsilon) \Rightarrow L-\varepsilon < f(x) < L+\varepsilon}$$

Consequently, $\boxed{L-\varepsilon < f(x)}$ (C)

Further it follows from (A) that $\boxed{0 < |x-a| < \delta_h(\varepsilon)}$

Since $\lim_{x \to a} h(x) = L$, condition (II) reads

$$\boxed{0 < |x-a| < \delta_h(\varepsilon) \Rightarrow L-\varepsilon < h(x) < L+\varepsilon}$$

From the last two green boxes we deduce

$$\boxed{h(x) < L+\varepsilon} \quad \text{(D)}$$

Now we summurize our findings: We assumed (A). Based on this assumption we proved:

$\boxed{f(x) \le g(x) \le h(x)}$ (B) and $\boxed{L-\varepsilon < f(x)}$ (C) and $\boxed{h(x) < L+\varepsilon}$ (D)

The transitivity of order and (B), (C) and (D) yield

$$\boxed{L-\varepsilon < f(x) \le g(x) \le h(x) < L+\varepsilon}$$

that is $\boxed{L-\varepsilon < g(x) < L+\varepsilon}$

the last green box is equivalent to $\boxed{|g(x)-L| < \varepsilon}$.

In conclusion (A) implies $|g(x)-L|<\varepsilon$. That is

we proved the implication:

$$0<|x-a|<\min\{\delta_f(\varepsilon), \delta_h(\varepsilon), \eta_0\} \Rightarrow |g(x)-L|<\varepsilon.$$

This proves $\lim_{x\to a} g(x) = L$.