Examples of Continuous Functions

Definition Let $D \subseteq \mathbb{R}$ be an interval.
A function $f : D \to \mathbb{R}$ is continuous on D if the following condition is satisfied:

$$\boxed{\forall c \in D \ \forall \varepsilon > 0} \ \exists \delta(\varepsilon, c) > 0 \text{ such that}$$

$$\forall x \in D \text{ we have } \boxed{|x-c| < \delta(\varepsilon, c)} \Rightarrow \boxed{|f(x)-f(c)| < \varepsilon}$$

Example Reipricol function

$$f(x) = \frac{1}{x} \text{ where } x \in (0, +\infty)$$

Here $D = (0, +\infty)$, $f(x) = 1/x$

Let $c > 0$ be arbitrary and let $\varepsilon > 0$ be arbitrary.

We have the following inequality

$(*)$ $\forall x \in \left(\frac{c}{2}, \frac{3c}{2}\right)$ we have $\boxed{\left|\frac{1}{x} - \frac{1}{c}\right| \leq \frac{2}{c^2}|x-c|}$

$\Leftrightarrow |x-c| < c/2$

Proof. Let $x \in \left(\frac{c}{2}, \frac{3c}{2}\right)$. Then $x > c/2 > 0$. Now simplify

$$\left|\frac{1}{x} - \frac{1}{c}\right| \underset{BK}{=} \left|\frac{c-x}{xc}\right| \underset{BK}{=} \frac{|x-c|}{xc} \underset{\text{Pizza-Party}}{\leq} \frac{|x-c|}{c^2/2} = \frac{2}{c^2}|x-c|.$$

$(*)$ Helps me find $\delta(\varepsilon, c) = \min\left\{\varepsilon \frac{c^2}{2}, \frac{c}{2}\right\}$

red = green

This is right coloring!

Now prove:

$$\boxed{\forall\, x>0}_{\,x\in D} \quad \boxed{|x-c|<\min\left\{\varepsilon\frac{c^2}{2}, \frac{c}{2}\right\}} \Rightarrow \boxed{\left|\frac{1}{x}-\frac{1}{c}\right|<\varepsilon}$$

Assume $\boxed{x>0}$ and $\boxed{|x-c|<\min\left\{\varepsilon\frac{c^2}{2}, \frac{c}{2}\right\}}$.

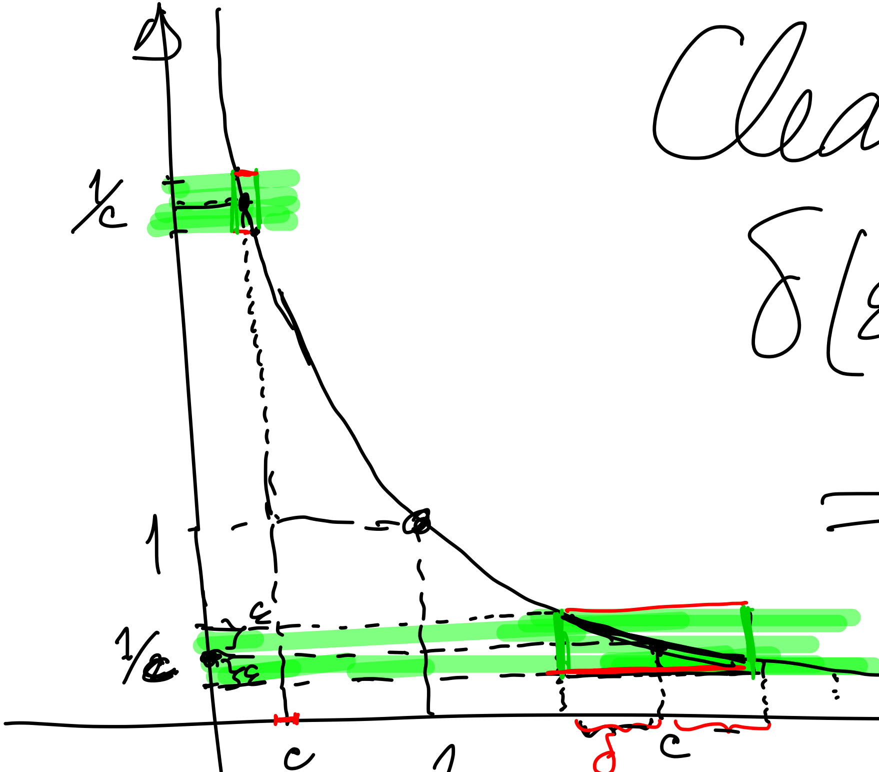
From this I deduce $\boxed{|x-c|<\frac{c}{2}}$ and $\boxed{\frac{2}{c^2}|x-c|<\varepsilon}$.

Hence $\circledast$ holds, that is $\boxed{\left|\frac{1}{x}-\frac{1}{c}\right|\leq\frac{2}{c^2}|x-c|}$

From the last two green boxes, I deduce by transitivity

$$\boxed{\left|\frac{1}{x}-\frac{1}{c}\right|<\varepsilon}$$

I greenified the red box. That is the PROOF.



Clearly here $\delta(\varepsilon, c)$ depends on c

thus the reciprocal function is NOT uniformly cont.

You can see that $1/x$ is UNIFORMLY CONTINUOUS on $[1, +\infty)$, but not on $(0,1]$

Example $\sin x$ and $\cos x$

Here $D = \mathbb{R}$. The key in the previous proof was the inequality

might or might not depend on c

$$\boxed{|f(x) - f(c)| \le K|x - c|}$$

Amazingly the following is true

$$\boxed{\forall x, c \in \mathbb{R} \quad |\sin x - \sin c| \le |x - c|}$$

$$\forall x, c \in \mathbb{R} \quad |\cos x - \cos c| \leq |x - c|$$

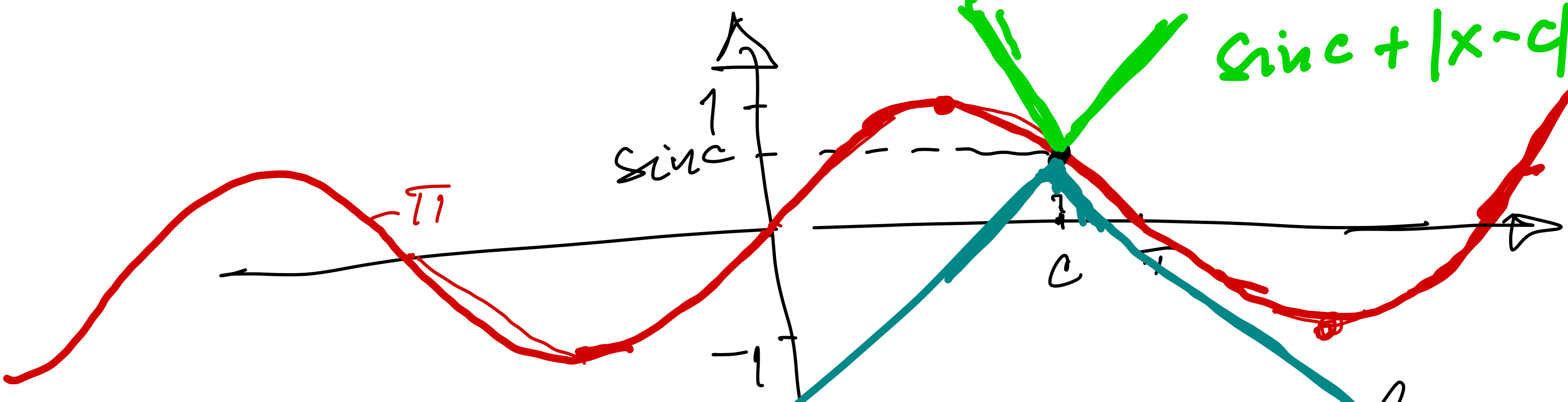
Before proving this I want to point out how this shows on the graph of $\sin x$ as a sandwich squeeze. Fix $c \in \mathbb{R}$.

[Bell, Burfla

$$|\underbrace{\sin x}_{\text{where is Waldo}} - \overset{\uparrow}{\underset{\text{Bell}}{\sin c}}| \leq \underbrace{|x-c|}_{\text{20 miles}}$$

$\Longleftrightarrow$

$$\underbrace{\boxed{\sin c - |x-c|}}_{\text{lower bun of a sadwich}} \leq \underbrace{\sin x}_{\text{ham}} \leq \underbrace{\boxed{\sin c + |x-c|}}_{\text{upper bun}}$$



I will post a nice picture! in fact a movie of this! and a proof on the unit circle ?

and it will be here.

A proof of:

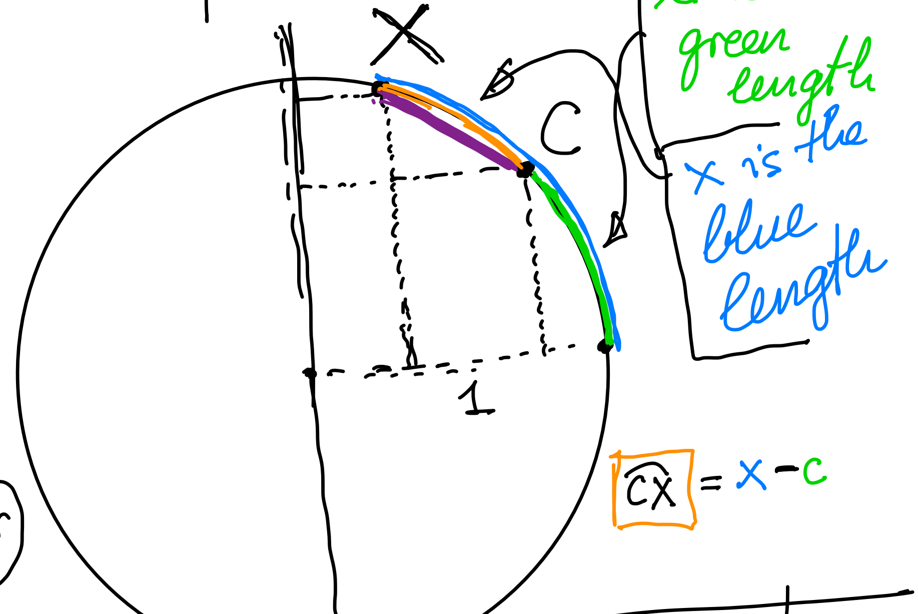
$\forall x, c \in \mathbb{R}$ $\quad |\cos x - \cos c| \le |x - c|$

and $|\sin x - \sin c| \le |x - c|$.

Let $X = (\cos x, \sin x)$

and $C = (\cos c, \sin c)$

X and C are points on the unit circle.



Compare the lengths $\boxed{\overline{CX}} \le \boxed{\overset{\frown}{CX}}$

straight line

smaller circular arc

The streight line is the shortest distance between two points. That is the basic property of a streight line.

$$\boxed{\overline{CX}} = \sqrt{(\cos x - \cos c)^2 + (\sin x - \sin c)^2} \geqslant$$

$$\geqslant \sqrt{(\cos x - \cos c)^2} = \left|\cos x - \cos c\right|$$

also $\geqslant \sqrt{(\sin x - \sin c)^2} = \left|\sin x - \sin c\right|$

Hence

$$\left|\cos x - \cos c\right| \leqslant \boxed{\overline{CX}} \leqslant \boxed{\overset{\frown}{CX}} \leqslant \left|x - c\right|$$

$$\left|\sin x - \sin c\right| \leqslant$$

smaller arc

This proves both $|\cos x - \cos c| \leq |x-c|$ and $|\sin x - \sin c| \leq |x-c|$

Proof that $\boxed{\widehat{CX}} \le |x-c|$.

Case 1. $|x-c| \le \pi$

Case 1a $0 < x-c \le \pi$. In this case to reach the point X one moves on the unit circle counterclockwise from the point C for the angle $x-c$ radians.

Therefore $\boxed{\widehat{CX}} = x-c = |x-c|$

Case 1b $0 < c-x \le \pi$. In this case to reach the point C one moves on the unit circle counterclockwise from the point X for the angle $c-x$ radians. Therefore

$\boxed{\widehat{CX}} = c-x = |x-c|$

Case 2 $|x-c| > \pi$. Since $\boxed{\overset{\frown}{cx}}$ is smaller of two circular arcs connecting c and x we always have $\boxed{\overset{\frown}{cx}} \leq \pi$. Thus $\boxed{\overset{\frown}{cx}} \leq |x-c|$.