Sequences

Definition A sequence is a function whose domain is either $\mathbb{N}$ or $\mathbb{N}_0$;
$\mathbb{N}$ is the set of all positive integers;
$\mathbb{N}_0$ is the set of all nonnegative integers;

$$\mathbb{N} = \{1, 2, 3, \ldots\}, \quad \mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$$

We will study the sequences of real numbers

$$s : \mathbb{N} \to \mathbb{R} \quad \text{or} \quad s : \mathbb{N}_0 \to \mathbb{R}$$

__Examples__ I $1, 2, 3, 4, \ldots;\ a_n = n$ for all $n \in \mathbb{N}$.

II $1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, \ldots$

$r_1 = 1,\ r_2 = 2,\ r_3 = 2,\ r_4 = 3,\ r_5 = 3,\ r_6 = 3,\ r_7 = 4, \ldots$

Is there a formula for r_n? Yes

$$r_n = \left\lfloor \frac{1}{2} + \sqrt{2n} \right\rfloor \text{ for all } n \in \mathbb{N}.$$

III $1, 2, 4, 8, 16, 32, 64, 128, 256, 512, \ldots$

Powers of two: $p_n = 2^n,\ n \in \mathbb{N}_0$

III For any nonzero real number $a \in \mathbb{R} \setminus \{0\}$ we have powers of a:

Here we use a recursive definition:

$$p_0 = 1, \quad \underbrace{p_n = a * p_{n-1}}_{\text{recursive formula}} \quad \forall n \in \mathbb{N}$$

$$p_1 = a * p_0 = a, \quad p_2 = a * a = a^2$$

$$p_3 = a * a^2 = a^3, \ldots$$

IV $\quad x_1 = 2, \quad x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}, \quad n \in \mathbb{N}$

$$x_2 = \frac{2}{2} + \frac{1}{2} = \frac{3}{2}, \quad x_3 = \frac{3}{4} + \frac{2}{3} = \frac{17}{12} \approx$$

$$x_4 = \frac{17}{24} + \frac{12}{17} \approx \qquad , \quad x_5 =$$

$x_n \to \sqrt{2}$ as $n \to +\infty$

Computers LOVE recursive formulas!

V $x_n = \left(1+\frac{1}{n}\right)^n, n \in \mathbb{N}$

$x_n \to e$ as $n \to +\infty$

relatated to P3A2

VI recursevily difined factorial:

$f_0 = 1, \quad f_n = n * f_{n-1}$ all $n \in \mathbb{N}$

$f_1 = 1 * f_0 = 1, \quad f_2 = 2 * 1, \quad f_3 = 3 * 2 * 1$

$f_4 = 4 * 3 * 2 * 1, \ldots, f_n = n * (n-1) * \cdots * 1$

$f_n = n!$ n factorial

$1, 1, 2, 6, 24, 120, 720, \ldots$ factorials

VII First we define a sequence of terms

$$t_n = \frac{1}{n!}, \quad n \in \mathbb{N}_0 .$$

Then we define the sequence of partial sums

$$s_0 = t_0, \quad s_n = s_{n-1} + t_n = s_{n-1} + \frac{1}{n!}$$

$$s_0 = \frac{1}{0!}, \quad s_1 = \frac{1}{0!} + \frac{1}{1!}, \quad s_2 = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!}$$

$$s_n = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{n!} = \sum_{k=0}^{n} \frac{1}{k!}$$

$$\longrightarrow e \quad (\text{as } n \to +\infty)$$

$$\sum_{k=0}^{+\infty} \frac{1}{k!} = e$$

Definition of the limit of sequence $L \in \mathbb{R}$

A sequence $s : \mathbb{N} \to \mathbb{R}$ has the limit L as $n \to +\infty$ if the following condition is satisfied:

$$\forall \varepsilon > 0 \ \exists N(\varepsilon) \in \mathbb{R} \text{ such that}$$

$$\forall n \in \mathbb{N} \quad n > N(\varepsilon) \Rightarrow |s_n - L| < \varepsilon$$

<u>Example</u> For $\forall r \in (-1,1)$ we have

$$\lim_{n \to +\infty} r^n = 0.$$

Proof. As with the limits $\lim_{x\to+\infty} f(x) = L$ we have to solve, for arbitrary $\varepsilon > 0$,

$$\underbrace{|r^n - 0|}_{\text{simplify: abs rules}} < \varepsilon$$

clearly $r \neq 0$ is only of interest

for $n \in \mathbb{N}_0$.

$$\underbrace{|r|^n < \varepsilon}_{\text{use ln to solve}}$$

ln is an increasing function: $\ln(|r|^n) < \ln(\varepsilon)$

$$\Updownarrow$$

$$n \ln(|r|) < \ln(\varepsilon)$$

this looks like a wrong direction

of inequality since we need a solution in the form $n > N(\varepsilon)$. However, $\ln(|r|) < 0$ since $0 \le |r| < 1$. Therefore multiplying by $\ln(|r|)$ will reverse the inequality: (OK)

The solution for n is

$$n > \frac{\ln(\varepsilon)}{\ln(|r|)} = N(\varepsilon)$$

Now prove, for arbitrary $\boxed{\varepsilon > 0}$ and $\boxed{r \in (-1,1), r \neq 0}$,

$$\forall n \in \mathbb{N} \quad \boxed{n > \frac{\ln(\varepsilon)}{\ln(|r|)}} \Rightarrow |r^n - 0| < \varepsilon$$

Let $n \in \mathbb{N}$ and assume $\boxed{n > \frac{\ln(\varepsilon)}{\ln(|r|)}}$.

Since $\boxed{r(-1,1) \text{ and } r \neq 0}$ we have $\boxed{|r| \in (0,1)}$. Hence $\boxed{\ln(|r|)<0}$.

Multiplying by we get $\boxed{n \ln(|r|) < \ln \varepsilon}$.

BK about the ln function: $\boxed{\ln(|r|^n) < \ln \varepsilon}$.

Since ln is an increasing function, we have $\boxed{|r|^n < \varepsilon}$. Now BK for the absolute value function yields $\boxed{|r^n| < \varepsilon}$. Consequently $\boxed{|r^n - 0| < \varepsilon}$.