

Theorems about Convergent Sequences

$f: \mathbb{N} \rightarrow \mathbb{R}$ or $f: \mathbb{N}_0 \rightarrow \mathbb{R}$

Definition of Limit

Let $L \in \mathbb{R}$. A sequence $s: \mathbb{N} \rightarrow \mathbb{R}$ has the limit L as $n \rightarrow +\infty$ if the following condition is satisfied :

$\forall \varepsilon > 0 \exists N(\varepsilon) \in \mathbb{R}$ such that

$\forall n \in \mathbb{N} \quad n > N(\varepsilon) \Rightarrow |s_n - L| < \varepsilon$

The notation for the limit is $\lim_{n \rightarrow +\infty} s_n = L$

Theorem Let $f: [1, +\infty) \rightarrow \mathbb{R}$ and define

the sequence $a: \mathbb{N} \rightarrow \mathbb{R}$ by

$$\forall n \in \mathbb{N} \quad a_n = f(n).$$

Then, if $\lim_{x \rightarrow +\infty} f(x) = L$, then $\lim_{n \rightarrow +\infty} a_n = L$.

Proof.

everything is green
in this definition

Here $D = [1, +\infty)$, $L \in \mathbb{R}$.

(I) $\exists X_0 \in D$ s.t. $[X_0, +\infty) \subseteq D$. Here $X_0 = 1$.

(II) $\forall \varepsilon > 0 \quad \exists X(\varepsilon) \geq 1$ such that

$$x > X(\varepsilon) \Rightarrow |f(x) - L| < \varepsilon.$$

Let $\varepsilon > 0$. What is $N(\varepsilon)$? We need:

$$\forall n \in \mathbb{N} \quad n > N(\varepsilon) \Rightarrow \left| a_n - L \right| < \varepsilon$$

must
yield

a_n
 $f(n)$

Set $N(\varepsilon) = X(\varepsilon)$.

Assume

$n \in \mathbb{N}$ and $n > N(\varepsilon)$. Then $n > X(\varepsilon)$.

By  $|f(n) - L| < \varepsilon$

By  the last green BOX can be
written as

$$|a_n - L| < \varepsilon$$

thus I proved :

$$\forall \varepsilon > 0 \exists N \text{ such that } n > N \Rightarrow |a_n - L| < \varepsilon .$$

By def. this proves $\lim_{n \rightarrow +\infty} a_n = L$.

Theorem Let $f: (0, 1] \rightarrow \mathbb{R}$ and let

$$\forall n \in \mathbb{N} \quad a_n = f(\frac{1}{n}).$$

If $\lim_{x \downarrow 0} f(x) = L$, then $\lim_{n \rightarrow +\infty} a_n = L$.

Theorem (Algebra of Limits) $K, L \in \mathbb{R}$.

Let $a: \mathbb{N} \rightarrow \mathbb{R}$, $b: \mathbb{N} \rightarrow \mathbb{R}$, $c: \mathbb{N} \rightarrow \mathbb{R}$ be sequences.

Assume $\lim_{n \rightarrow +\infty} a_n = K$ and $\lim_{n \rightarrow +\infty} b_n = L$.

- A If $\forall n \in \mathbb{N} c_n = a_n + b_n$, then $\lim_{n \rightarrow \infty} c_n = K + L$
- B If $\forall n \in \mathbb{N} c_n = a_n \cdot b_n$, then $\lim_{n \rightarrow \infty} c_n = K \cdot L$
- C If $\forall n \in \mathbb{N}, b_n \neq 0$ and $c_n = \frac{a_n}{b_n}$ and $L \neq 0$,
 then $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{K}{L}$.

Proof. more it!

- 1 means $\forall \varepsilon > 0 \exists N_a(\varepsilon) \in \mathbb{R}$ such that
 $\forall n \in \mathbb{N} n > N_a(\varepsilon) \Rightarrow |a_n - K| < \varepsilon$

2

means $\forall \varepsilon > 0 \exists N_b(\varepsilon) \in \mathbb{R}$ such that

$$\forall n \in \mathbb{N} \quad n > N_b(\varepsilon) \nrightarrow |b_n - L| < \varepsilon$$

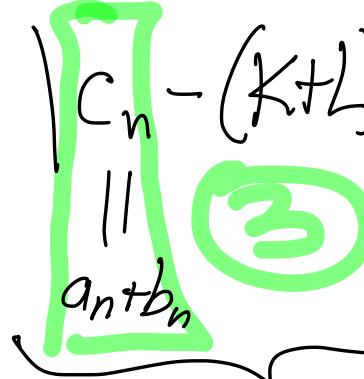


4

demands

$\forall \varepsilon > 0 \exists N_c(\varepsilon)$ such that

$$\forall n \in \mathbb{N} \quad n > N_c(\varepsilon) \nrightarrow |c_n - (k+l)| < \varepsilon$$



3

study this
and connect it to

$$|a_n + b_n - (K+L)|$$

$$= |a_n + b_n - K - L| = |a_n - K + b_n - L|$$

by triangle inequality



$$\leq |a_n - K| + |b_n - L|$$



⋮

