Infinite Series

We are given a sequence, in this case reciprocals of the factorials

$\frac{1}{0!}, \frac{1}{1!}, \frac{1}{2!}, \dots, \frac{1}{n!}, \dots$ Sequence $\left\{\frac{1}{n!}\right\}$

Then we from the partial sums:

$S_0 = \frac{1}{0!}$

$S_1 = \frac{1}{0!} + \frac{1}{1!}$

$S_2 =$

$S_3 =$

$\vdots$

$S_n = \frac{1}{0!} + \dots + \frac{1}{n!} = \sum_{k=0}^{n} \frac{1}{k!}$

$\vdots$

This is a new sequence, called the sequence of PARTIAL SUMS $\{S_n\}$

Then we ask whether the sequence $\{S_n\}$ converges? We answered, Yes, and we write
$$\sum_{k=0}^{\infty} \frac{1}{k!} = e$$
$$e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{n!} + \dots$$
indicating an infinite series

In general, given a sequence $\{a_n\}$, that is $a_1, a_2, \dots, a_n, \dots$ we form a new sequence

$S_1 = a_1$

$S_2 = a_1 + a_2$

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$S_n = \sum_{k=1}^{n} a_k$, and we

whether the sequence $\{S_n\}$, of partial sums converges

We introduce a symbol

$$\sum_{n=1}^{\infty} a_n$$ and call it an INFINITE SERIES

a_n are the terms of $\sum_{n=1}^{\infty} a_n$

If the sequence $\{S_n\}$ of partial sums converges we say that the series $\sum_{n=1}^{\infty} a_n$ CONVERGES

Otherwise we say that the series $\sum_{n=1}^{\infty} a_n$ diverges.

Example Let $a_n = \left(\frac{1}{2}\right)^n$ with $n \in \mathbb{N}$.

Powers of $1/2$

Then we form the partial sums

$S_1 = \frac{1}{2}$

$S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = 7/8$

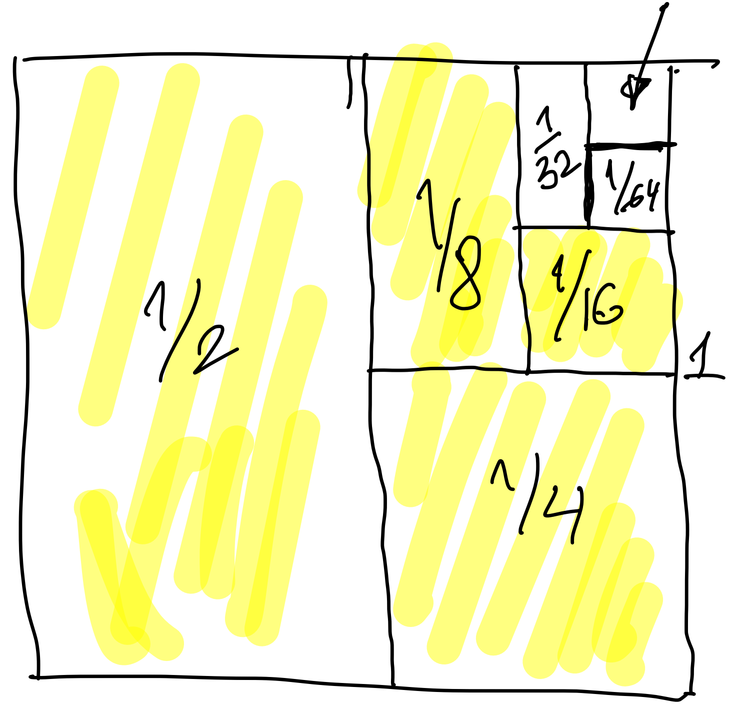
$S_4 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = 15/16$

$\vdots$

$S_n = \sum_{k=1}^{n} \left(\frac{1}{2}\right)^k = 1 - \frac{1}{2^n}$

$\vdots$

$\lim_{n \to \infty} S_n = 1$



In the language of infinite series:

The infinite series $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \underbrace{1}_{\text{the sum of infinite series}}$ converges to 1

Example $a_n = (-1)^n$ with $n \in \mathbb{N}$. Powers of (-1)

$S_1 = -1$

$S_2 = -1+1 = 0$

$S_3 = -1+1-1 = -1$

$S_4 = -1+1-1+1 = 0$

$\vdots$

$$S_n = \sum_{k=1}^{n} (-1)^k = \frac{1}{2}\left(-1+(-1)^n\right)$$

Clearly the sequence $\{S_n\}$ does not converge.

The infinite series $\sum_{n=1}^{\infty} (-1)^n$ diverges.

Example $a_n = n$ with $n \in \mathbb{N}$.

$S_1 = 1$

$$S_2 = 1+2 = 3$$

$$S_3 = 1+2+3 = 6$$

$$S_4 = 1+2+3+4 = 10$$

$$\vdots$$

$$S_n = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

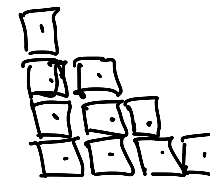
$$1+2+\dots+100 = S_{100} = 5050$$

$$100+99+\dots+1 = S_{100}$$

$$101+101+\dots+101 = 2S_{100}$$

$$\frac{100 * 101}{2} = S_{100}$$

triangular numbers



$$\lim_{n\to+\infty} S_n = +\infty$$

The series $\sum_{n=1}^{\infty} n$ diverges

Definition An infinite series whose terms are constant mult. of powers of a real number is called a **GEOMETRIC SERIES**:

$$\sum_{n=0}^{\infty} ar^n$$

a ← constant

If $a = 0$, then all the terms are equal to 0, so the sum is 0. (Not interesting)

So, we assume $a \neq 0$.

$$S_n = a + \cancel{ar} + \cancel{ar^2} + \cdots + \cancel{ar^n}$$

$$rS_n = \cancel{ar} + \cancel{ar^2} + \cancel{ar^3} + \cdots + \cancel{ar^n} + ar^{n+1}$$

to get a simple formula for S_n

Subtract

$$S_n - rS_n = a - ar^{n+1}$$

$$(1-r)S_n = a(1-r^{n+1})$$

assume $r \neq 1$

$$\boxed{S_n = a\frac{1-r^{n+1}}{1-r}}$$

True for all $n \in \mathbb{N} \cup \{0\}$ all $r \in \mathbb{R} \setminus \{1\}$ and all $a \in \mathbb{R}$

$r = 1$ (is a problem

$S_n = na$

$\lim_{n \to +\infty} S_n$? Does it converge or NOT ?

This depends on $\lim_{n \to +\infty} r^n$? Does this converge or not

We proved that $\boxed{\forall r \in (-1,1) \quad \lim_{n \to +\infty} r^n = 0}$

We could prove $\forall r \notin (-1,1]$ $\lim_{n \to +\infty} r^n$ Does Not Exist.

Algebra of Limits yields

$\forall r \in (-1,1)$ we have

$|r|<1$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} a\frac{1-r^{n+1}}{1-r} = \frac{a}{1-r}$$

In the language of Infinite Series:

$$\forall r \in (-1,1) \quad \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

$\forall r \notin (-1,1)$ $\forall a \in \mathbb{R}\setminus\{0\}$ $\sum_{n=0}^{\infty} ar^n$ <u>diverges</u>